Quarks and Fractionally Quantized Vortices in Superconductors
- An analogy between two worlds -

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There are profound analogies between ideas in particle physics and the corresponding ones in condensed-matter physics. There is also an interesting analogy between quarks and fractionally quantized-flux vortices in superconductors. Half-quantized flux vortices exist in two-component superconductors under a magnetic field. They can be interpreted as a monopole, and two half-quantum flux vortices form a bound state connected by a domain wall. We can formulate vortices with 1/3-quantum flux in three-component superconductors with equivalent three bands. Under the duality transformation between charge and magnetic flux, vortices with 1/3-quantum magnetic flux can be regarded as quarks with the 1/3 charge. A three-vortex bound state is formulated and this bound state resembles a baryon in QCD. We have also a two-vortex bound state, formed by vortices with 1/3 and -1/3 quantized flux, respectively, that corresponds to a meson. Two or three fractionally quantized vortices are connected by domain walls and form a bound state. The quark confinement is discussed by analogy with baryonic bound states in superconductors

I. INTRODUCTION

There is an interesting and profound analogy between particles physics and superconductivity. This was first pointed out by Youichiro Nambu, and he invented a concept of spontaneous symmetry breaking in particles physics. We say that spontaneous symmetry breaking occurred when the ground state of the system lost the invariance of the Hamiltonian of the system. There is no reason why an invariance of the Hamiltonian should be an invariance of the ground state. A good basic example is the Heisenberg model with spin-spin exchange interactions between nearest neighbors. The ground state is not rotationally invariant, with spins aligned in one direction, although the Hamiltonian is rotationally invariant. The ground state has a long-range order by breaking the rotational invariance. It is well known that the gapless Goldstone mode exists when the continuous symmetry is spontaneously broken. In the Heisenberg ferromagnet, this gapless mode is the spin wave excitation. There are many models that exhibit spontaneous symmetry breaking in the condensed-matter physics.

Superconductivity is most familiar phenomenon that occurs as a result of spontaneous symmetry breaking. This is described by the Ginzburg-Landau functional[1]:

$$F = \int d^3x \Big[\alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{\hbar^2}{2m}\Big|\Big(\nabla + i\frac{2\pi}{\phi_0}\mathbf{A}\Big)\psi\Big|^2 + \frac{1}{8\pi}H^2\Big]. \tag{1}$$

ψ is the order parameter, $\beta > 0$ and α is written as $\alpha = \alpha_0(T - T_c)$. This free energy describes a spontaneous breaking of $U(1)$ symmetry. When $\alpha < 0$, that is $T < T_c$, the minimization of F yields a solution $\psi \neq 0$. The potential

$$V(\psi) = \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4, \tag{2}$$

has an infinite number of possible minima for $\alpha < 0$ given by

$$\psi = \sqrt{-\frac{\alpha}{\beta}}e^{i\theta} \tag{3}$$

for any real angle θ in the range $0 \le \theta \le 2\pi$. Any choice of θ would have exactly the same energy that implies the existence of a massless Nambu-Goldstone boson.

There is a similarity between the Dirac equation and the gap equation of superconductivity. Nambu first noticed this fact[2]. The Dirac equation is written as

$$\left[p_0 - \begin{pmatrix} \mathbf{p}\cdot\sigma & 0 \\ 0 & -\mathbf{p}\cdot\sigma \end{pmatrix} - \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}\right]\begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = 0. \tag{4}$$

Here σ_j ($j = 1, 2, 3$) are Pauli matrices. If the mass m vanishes, the Dirac equation possesses chiral symmetry and is invariant under the transformation

$$\psi \to e^{i\alpha\gamma_5}\psi, \tag{5}$$

for $\psi = (\psi_R, \psi_L)^t$ and

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{6}$$

The mass generation occurs as a result of spontaneous breaking of chiral symmetry[3, 4]. The mass is determined by the self-consistency equation (Fig.1). This equation has a close analogy with superconductivity; the gap is generated by spontaneous breaking of $U(1)$ symmetry[5, 6]. The equation for paired electrons is written as

$$\left[p_0 - \begin{pmatrix} \xi_p & 0 \\ 0 & -\xi_p \end{pmatrix} - \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}\right]\begin{pmatrix} \psi_{p\uparrow} \\ \psi^\dagger_{-p\downarrow} \end{pmatrix} = 0, \tag{7}$$

where ξ_p is the kinetic energy measured from the Fermi surface and Δ is the energy gap. This equation gives the eigenvalues

$$p_0 = E_p = \pm\sqrt{\xi_p^2 + \Delta^2}. \tag{8}$$

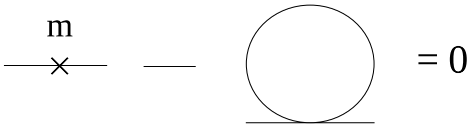

FIG. 1: Self-consistency equation for the mass.

TABLE I:

Superconductors	Particles
Broken symmetry	Broken symmetry
Ground state: BCS	Vacuum
Energy gap	Mass
Quasi particles	Particles
Gapless mode (Anderson-Bogoliubov)	Nambu-Goldstone mode
Plasma mode	Higgs mechanism
Meissner effect	Higgs mechanism
TRS breaking	CP violation
Half-quantum vortex	Monopole
Fractional vortices	Quarks
Leggett mode	?

Therefore there is a close analogy between the Dirac equation in elementary particle theory and the gap equation in the theory of superconductivity.

We show a list of correspondences between superconductivity and particle physics in Table I. The gapless mode emerging from the spontaneously broken global U(1) symmetry in superconductors is known as the Anderson-Bogoliubov mode. TRS breaking denotes time-reversal symmetry breaking which occurs in a triplet-pairing superconductor or in a three-band superconductor.

II. MASS AND GAP EQUATIONS

We discuss a similarity between superconductivity and particle physics in the gap equation.

A. Superconductivity

Let us consider the BCS Hamiltonian:

$$\begin{aligned} H &= \int d\mathbf{r} \sum_\sigma \psi_\sigma^\dagger(\mathbf{r}) \left(\frac{\mathbf{p}^2}{2m} - \mu \right) \psi_\sigma(\mathbf{r}) \\ &- g \int d\mathbf{r} \psi_\uparrow^\dagger(\mathbf{r}) \psi_\downarrow^\dagger(\mathbf{r}) \psi_\downarrow(\mathbf{r}) \psi_\uparrow(\mathbf{r}), \end{aligned} \tag{9}$$

where σ is the spin index $\uparrow$ and $\downarrow$, μ is the chemical potential and $g > 0$ is the coupling constant of the attractive interaction. In the momentum space, this is written as

$$H = \sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} - g \frac{1}{V} \sum_{kk'q} c_{k'\uparrow}^\dagger c_{-k'+q\downarrow}^\dagger c_{-k+q\downarrow} c_{k\uparrow}, \tag{10}$$

where $\xi_k = \epsilon_k - \mu$ for the electron dispersion ϵ_k. The corresponding Lagrangian density is

$$\begin{aligned} \mathcal{L} &= \sum_\sigma \psi_\sigma^\dagger(x) \left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} + \mu \nabla^2 \right) \psi_\sigma(x) \\ &+ g \psi_\uparrow^\dagger(x) \psi_\downarrow^\dagger(x) \psi_\downarrow(x) \psi_\uparrow(x). \end{aligned} \tag{11}$$

Using the Nambu notation[7],

$$\psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow^\dagger \end{pmatrix}, \tag{12}$$

the Lagrangian density becomes

$$\mathcal{L} = \psi^\dagger \left(\sigma_0 i\hbar \frac{\partial}{\partial t} - \sigma_3 \xi(\nabla) \right) \psi - \frac{g}{4} \left[(\psi^\dagger \psi)^2 - (\psi^\dagger \sigma_3 \psi)^2 \right], \tag{13}$$

where σ_0 is the unit matrix and $\xi(\nabla) = -\hbar^2 \nabla^2/(2m) - \mu = \mathbf{p}^2/(2m) - \mu$. The vacuum partition function is represented by a functional integral,

$$Z = \int d\psi^\dagger d\psi \exp\left(\frac{i}{\hbar} \int d^d x \mathcal{L} \right). \tag{14}$$

d is the space-time dimension. This can be written in a bilinear form by applying a Hubbard-Stratonovich transformation,

$$\begin{aligned} &\exp\left(\frac{i}{\hbar} g \int d^d x \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow \right) = \\ &\int d\Delta^* d\Delta \exp\left[-\frac{i}{\hbar} \int d^d x \left(\Delta^* \psi_\downarrow \psi_\uparrow + \Delta \psi_\uparrow^\dagger \psi_\downarrow^\dagger + \frac{1}{g} |\Delta|^2 \right) \right], \end{aligned} \tag{15}$$

where Δ^* and Δ are auxiliary fields and an overall normalization factor is excluded. The partition function has the form

$$Z = \int d\psi^\dagger d\psi \int d\Delta^* d\Delta \exp\left(\frac{i}{\hbar} \int d^d x \mathcal{L}_{eff} \right), \tag{16}$$

where

$$\mathcal{L}_{eff} = \psi^\dagger \left[\sigma_0 i\hbar \frac{\partial}{\partial t} - \sigma_3 \xi(\nabla) - \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix} \right] \psi - \frac{1}{g} |\Delta|^2. \tag{17}$$

The field equations obtained by variation of the Lagrangian are

$$\left[i\hbar \frac{\partial}{\partial t} - \sigma_3 \xi(\nabla) - \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix} \right] \psi = 0, \tag{18}$$

$$\Delta = g \psi_\uparrow \psi_\downarrow. \tag{19}$$

The equation for Δ shows that Δ describes a pair of electrons that forms a spin-singlet. If we approximate Δ by its average $\bar{\Delta} = g\langle\psi_\uparrow\psi_\downarrow\rangle$, we obtain a self-consistency equation for $\bar{\Delta}$. By performing the Grassman integration over the fields $\psi^\dagger$ and ψ, we obtain the effective action

$$S(\Delta^*, \Delta) = -\frac{1}{g}\int d^d x|\Delta(x)|^2 - i\hbar \mathrm{Tr}\ln\begin{pmatrix} p_0 - \xi(\mathbf{p}) & -\Delta(x) \\ -\Delta^*(x) & p_0 + \xi(\mathbf{p}) \end{pmatrix}, \quad (20)$$

for which the partition function is

$$Z = \int d\Delta^* d\Delta \exp\left(\frac{i}{\hbar}S(\Delta^*, \Delta)\right). \quad (21)$$

Now the averaged value $\bar{\Delta}$ of the gap function Δ is determined by adopting the saddle point approximation. The field equation reads

$$\frac{\delta S(\bar{\Delta}^*, \bar{\Delta})}{\delta \bar{\Delta}^*} = 0. \quad (22)$$

We obtain a solution assuming that $\bar{\Delta} > 0$ is a constant. This yields

$$\frac{1}{g}\bar{\Delta} = i\hbar \mathrm{Tr} G_0(p)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (23)$$

where $G_0(p)$ is the Green function including $\bar{\Delta}$,

$$G_0(p) = \begin{pmatrix} p_0 - \xi(\mathbf{p}) & -\bar{\Delta} \\ -\bar{\Delta}^* & p_0 + \xi(\mathbf{p}) \end{pmatrix}^{-1} = \frac{1}{p_0^2 - E(\mathbf{p})^2 + i\delta}\begin{pmatrix} p_0 + \xi(\mathbf{p}) & \bar{\Delta} \\ \bar{\Delta}^* & p_0 - \xi(\mathbf{p}) \end{pmatrix}. \quad (24)$$

Here,

$$E(\mathbf{p}) = \sqrt{\xi(\mathbf{p})^2 + \bar{\Delta}^2} \quad (25)$$

is the single-particle excitation energy. Then we obtain the gap equation

$$\frac{1}{g} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{E(\mathbf{k})}. \quad (26)$$

The superconducting gap is

$$\bar{\Delta} = 2\hbar\omega_D \exp\left(-\frac{1}{\rho g}\right), \quad (27)$$

with the energy cutoff $\hbar\omega_D$ and the density of states ρ at the Fermi energy.

B. Nambu-Jona-Lasinio Model

The Nambu-Jona-Lasinio model is

$$\mathcal{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi + g[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2], \quad (28)$$

which has the form similar to the BCS model. We set $\hbar = 1$ in this section. γ_μ and γ_5 are Dirac gamma matrices. This Lagrangian is invariant under the particle number and chiral transformations,

$$\psi \to \exp(i\alpha)\psi, \quad \bar{\psi} \to \bar{\psi}\exp(-i\alpha) \quad (29)$$
$$\psi \to \exp(i\gamma_5\alpha)\psi, \quad \bar{\psi} \to \bar{\psi}\exp(i\gamma_5\alpha). \quad (30)$$

In a similar way after the spontaneous symmetry breaking, the fermion (nucleon) acquires a mass $m \propto 2g\langle\bar{\psi}\psi\rangle$.

Using an identity

$$1 = \text{const.}\int d\sigma' d\pi' \exp i\int d^4x\Big[-\frac{1}{4g}(\sigma'^2 + \pi'^2)\Big], \quad (31)$$

the partition function is written as

$$Z = \int d\bar{\psi} d\psi d\sigma' d\pi' \exp i\int d^4x\Big[\bar{\psi} i\gamma^\mu\partial_\mu\psi + g((\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2) - \frac{1}{4g}(\sigma'^2 + \pi'^2)\Big]. \quad (32)$$

We define new σ and π fields by

$$\sigma' = \sigma + 2g\bar{\psi}\psi, \quad (33)$$
$$\pi' = \pi + 2gi\bar{\psi}\gamma_5\psi, \quad (34)$$

then we have

$$Z = \int \bar{\psi} d\psi d\sigma d\pi \exp\left(i\int d^4x \mathcal{L}_{eff}\right), \quad (35)$$

where

$$\mathcal{L}_{eff} = \mathcal{L} - \frac{1}{4g}[(\sigma + 2g\bar{\psi}\psi)^2 + (\pi + 2gi\bar{\psi}\gamma_5\psi)^2] = \bar{\psi}[i\gamma^\mu\partial_\mu - (\sigma + i\gamma_5\pi)]\psi - \frac{1}{4g}(\sigma^2 + \pi^2). \quad (36)$$

Then we obtain the effective action

$$S_{eff}(\sigma, \pi) = -i\ln\det[i\gamma^\mu\partial_\mu - (\sigma + i\gamma_5\pi)] - \frac{1}{4g}\int d^4x(\sigma^2 + \pi^2) = -i\mathrm{Tr}\ln[i\gamma^\mu\partial_\mu - (\sigma + i\gamma_5\pi)] - \frac{1}{4g}\int d^4x(\sigma^2 + \pi^2). \quad (37)$$

The saddle point approximation leads to a solution such that $\sigma = \sigma_0$ is a constant and $\pi = 0$. The equation for σ_0 is

$$\sigma_0 = 2ig\mathrm{Tr}\frac{1}{i\gamma^\mu\partial_\mu - \sigma_0}. \quad (38)$$

Because σ_0 is the mass m of the fermion ψ, the mass m is determined by

$$1 = 8gi\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - m^2} = 8g\int \frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^2 + m^2}, \quad (39)$$

which has a nontrivial solution $m \neq 0$ when

$$0 < \frac{2\pi^2}{g\Lambda^2} < 1. \quad (40)$$

In this case the mass m is determined by

$$\frac{2\pi^2}{g\Lambda^2} = 1 - \frac{m^2}{\Lambda^2} \ln\left(1 + \frac{\Lambda^2}{m^2}\right). \quad (41)$$

We define the field h_σ by

$$\sigma = \sigma_0 + h_\sigma, \quad (42)$$

then h_σ is a massive boson whose mass is $2m$. The field π represents a massless boson which is the Nambu-Goldstone boson. The result that the field h_σ acquires the mass $2m$ is well understood by an analogy to superconductivity, where the excitation energy is $2\bar{\Delta}$ when a pair of electrons is excited above the Fermi energy. The model has been generalized to a more realistic two-flavor model:

$$\mathcal{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi + g[(\bar{\psi}\psi)^2 - \sum_i (\bar{\psi}\gamma_5\tau_i\psi)(\bar{\psi}\gamma_5\tau_i\psi)]. \quad (43)$$

As is obvious from the discussion here, the nucleon mass generation is very analogous to the gap generation in superconductors. We note that the mass is finite only when $0 < 2\pi^2/g\Lambda^2 < 1$ holds in the Nambu-Jona-Lasinio model while the superconducting gap always exists as far as $g > 0$.

III. MULTI-BAND SUPERCONDUCTIVITY

We consider the multi-band BCS model with the attractive interactions:

$$\begin{aligned} H &= \sum_{i\sigma} \int d\mathbf{r} \psi_{i\sigma}^\dagger(\mathbf{r}) K_i(\mathbf{r}) \psi_{i\sigma}(\mathbf{r}) \\ &- \sum_{ij} g_{ij} \int d\mathbf{r} \psi_{i\uparrow}^\dagger(\mathbf{r}) \psi_{i\downarrow}^\dagger(\mathbf{r}) \psi_{j\downarrow}(\mathbf{r}) \psi_{j\uparrow}(\mathbf{r}), \end{aligned} \quad (44)$$

where i and j (=1,2,$\cdots$) are band indices. $K_i(\mathbf{r})$ stands for the kinetic operator. We assume that $g_{ij} = g_{ji}^*$. The second term is the pairing interaction and g_{ij} are coupling constants. The mean-field Hamiltonian is

$$\begin{aligned} H_{MF} &= \sum_i \int d\mathbf{r} \Big[\sum_\sigma \psi_{i\sigma}^\dagger(\mathbf{r}) K_i(\mathbf{r}) \psi_{i\sigma}(\mathbf{r}) \\ &+ \Delta_i(\mathbf{r}) \psi_{i\uparrow}^\dagger(\mathbf{r}) \psi_{i\downarrow}^\dagger(\mathbf{r}) + \Delta_i^*(\mathbf{r}) \psi_{i\downarrow}(\mathbf{r}) \psi_{i\uparrow}(\mathbf{r}) \Big], \end{aligned} \quad (45)$$

where the gap function in each band is defined by

$$\Delta_i(\mathbf{r}) = -\sum_j g_{ij} \langle \psi_{j\downarrow}(\mathbf{r}) \psi_{j\uparrow}(\mathbf{r}) \rangle, \quad (46)$$

and its complex conjugate is

$$\Delta_i^*(\mathbf{r}) = -\sum_j g_{ji} \langle \psi_{j\uparrow}^\dagger(\mathbf{r}) \psi_{j\downarrow}^\dagger(\mathbf{r}) \rangle. \quad (47)$$

We define Green's functions as follows[8],

$$G_{j\sigma\sigma'}(x - x') = -\langle T_\tau \psi_{j\sigma}(x) \psi_{j\sigma'}^\dagger(x') \rangle, \quad (48)$$

$$F_{j\sigma\sigma'}^+(x - x') = \langle T_\tau \psi_{j\sigma}^\dagger(x) \psi_{j\sigma'}^\dagger(x') \rangle, \quad (49)$$

where T_τ is the time-ordering operator and we use the notation $x = (\tau, \mathbf{r})$. In terms of the Green's functions, the gap functions satisfy the system of equations

$$\begin{aligned} \Delta_i^*(\mathbf{r}) &= \sum_j g_{ij}^* F_{j\downarrow\uparrow}^+(\tau' = \tau + 0; \mathbf{r}, \mathbf{r}) \\ &= \sum_j g_{ij}^* \frac{1}{\beta} \sum_n F_{j\downarrow\uparrow}^+(i\omega_n; \mathbf{r}, \mathbf{r}). \end{aligned} \quad (50)$$

This yields the gap equation,

$$\Delta_i = \sum_j g_{ij} N_j \Delta_j \int d\xi_j \frac{1}{E_j} \tanh\left(\frac{E_j}{2T}\right), \quad (51)$$

where $E_j = \sqrt{\xi_j^2 + |\Delta_j|^2}$ and T is the temperature where we set Boltzmann constant k_B to unity. N_j is the density of states at the Fermi surface. Since all the bands couple with each other through mutual interactions g_{ij}, we have one critical temperature T_c[9, 10]. At the critical temperature $T = T_c$, this equation reads[11]1

$$\Delta_i = \ln\left(\frac{2e^\gamma \omega_c}{\pi T_c}\right) \sum_j g_{ij} N_j \Delta_j, \quad (52)$$

for the cutoff energy ω_c. γ denotes the Euler constant. Here we assume the same cutoff energy in all the interactions. The system of equations in eq.(50) yields a set of differential equations[11]

$$\begin{aligned} \Delta_j^*(\mathbf{r}) &= \ln\left(\frac{2e^\gamma \omega_c}{\pi T}\right) \sum_\ell g_{j\ell}^* N_\ell \Delta_\ell^*(\mathbf{r}) \\ &+ \frac{7\zeta(3)}{48(\pi T_c)^2} \sum_\ell g_{j\ell}^* N_\ell v_\ell^2 \left(\nabla + i\frac{2e}{\hbar c}\mathbf{A}\right)^2 \Delta_\ell^*(\mathbf{r}) \\ &- \frac{7\zeta(3)}{8(\pi T_c)^2} \sum_\ell g_{j\ell}^* N_\ell \Delta_\ell^*(\mathbf{r}) |\Delta_\ell(\mathbf{r})|^2. \end{aligned} \quad (53)$$

Here, e is the charge of the electron, and v_ℓ is the electron velocity at the Fermi surface in the ℓ-th band. The Planck constant $\hbar$ has been dropped except in front of the vector potential $\mathbf{A}$. We set $a_{mn} = g_{mn} N_n$. Then, the equation for Δ_j is

$$\left[a_{jj} \ln\left(\frac{2e^\gamma \omega_c}{\pi T}\right) - 1\right] \Delta_j + \ln\left(\frac{2e^\gamma \omega_c}{\pi T}\right) \sum_{\ell(\neq j)} a_{j\ell} \Delta_\ell + \cdots = 0. \quad (54)$$

In the second term of the right-hand side we can replace T by T_c near the transition temperature. We define the matrix $A=(a_{ij})$ and its inverse $A^{-1}=(b_{j\ell})$. Then the equation for Δ_j reads

$$\left[a_{jj}\ln\left(\frac{2e^{\gamma}\omega_c}{\pi T}\right)-1\right]\Delta_j+\sum_{\ell(\neq j)}a_{j\ell}\sum_m b_{\ell m}\Delta_m+\cdots=0. \tag{55}$$

For example, for $j=1$ we obtain

$$g_{11}\Big[\left(N_1\ln\left(\frac{2e^{\gamma}\omega_c}{\pi T}\right)-\frac{1}{\det G}(G^{-1})_{11}\right)\Delta_1 - \frac{1}{\det G}(G^{-1})_{12}\Delta_2-\frac{1}{\det G}(G^{-1})_{13}\Delta_3\Big]+\cdots=0. \tag{56}$$

where $G=(g_{ij})$ is the matrix of coupling constants. To obtain the multi-band Ginzburg-Landau functional, we multiply eq.(55) by $\Delta_j^*N_j$ and take a summation with respect to j. We use the gap equation $\Delta_\ell^*=\eta\sum_j g_{\ell j}^*N_j\Delta_j^*$ at $T=T_c$. Then the energy functional density f is

$$\begin{aligned} f &= -\sum_j\left(N_j\ln\frac{2e^{\gamma}\omega_c}{\pi T}-(G^{-1})_{jj}\right)|\Delta_j|^2 \\ &+ \sum_{j\ell}\Delta_j^*(G^{-1})_{j\ell}\Delta_\ell \\ &- \frac{7\zeta(3)}{48\pi^2T_c^2}\sum_\ell N_\ell v_\ell^2\Delta_\ell^*\left(\nabla-i\frac{2e}{\hbar c}\mathbf{A}\right)^2\Delta_\ell \\ &+ \frac{7\zeta(3)}{16\pi^2T_c^2}\sum_\ell N_\ell|\Delta_\ell|^4. \end{aligned} \tag{57}$$

Here we neglected unimportant constants. The fourth order term is simply given by $(|\Delta_\ell|^2)^2$. We can also derive this functional using the functional integral method.

IV. ABELIAN PROJECTION AND A SINGULARITY[12]

Let us consider a field $X(x)$. We assume that $X(x)$ is hermitian and is transformed as follows.

$$X(x)\to U(x)X(x)U(x)^{\dagger}. \tag{58}$$

We adopt that $X(x)$ is diagonalized:

$$X(x)=\begin{pmatrix}\lambda_1(x) & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \lambda_N(x)\end{pmatrix}, \tag{59}$$

where $\lambda_j(x)$ $(j=1,\cdots,N)$ are eigenvalues of $X(x)$. If all the $\lambda_j(x)$ are different each other, $X(x)$ is invariant under the transformation $X(x)\to U(x)X(x)U(x)^{\dagger}$, where

$$U(x)=\begin{pmatrix}e^{i\theta_1(x)} & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & e^{i\theta_N(x)}\end{pmatrix}. \tag{60}$$

Here $\sum_{i=1}^N\theta_i(x)=0$ and we assume that $[X(x),U(x)]=0$. $U(x)$ is an element of $U(1)^{N-1}$.

Let us assume that two of $\lambda_j(x)$ are degenerate, for example, $\lambda_i(x)=\lambda_j(x)$ for $i\neq j$. Then we have $SU(2)$ symmetry at this point. It is known that a monopole-like singularity appears at the point where the eigenvalues are degenerate. The same singularity appears in a vortex state with fractionally quantized flux in a multi-band superconductor. Let us consider the $SU(2)$ case. The matrix $U(x)$ is written as, using Pauli matrices σ_j $(j=1,2,3)$,

$$\begin{aligned} U(x) &= e^{i\xi(x)\sigma_3/2}e^{i\theta(x)\sigma_2/2}e^{i\varphi(x)\sigma_3/2} \\ &= \begin{pmatrix} e^{i(\varphi+\xi)(x)/2}\cos\frac{\theta(x)}{2} & e^{-i(\varphi-\xi)(x)/2}\sin\frac{\theta(x)}{2} \\ -e^{i(\varphi-\xi)(x)/2}\sin\frac{\theta(x)}{2} & e^{-i(\varphi+\xi)(x)/2}\cos\frac{\theta(x)}{2}\end{pmatrix}, \end{aligned} \tag{61}$$

where ξ, θ and φ are angles in the range of $0\le\xi\le 2\pi$, $0\le\theta\le\pi$ and $0\le\varphi\le 2\pi$. θ and φ are Euler angles. $U^{\dagger}(x)$ is given by

$$U^{\dagger}(x)=\begin{pmatrix} e^{-i(\varphi+\xi)(x)/2}\cos\frac{\theta(x)}{2} & -e^{-i(\varphi-\xi)(x)/2}\sin\frac{\theta(x)}{2} \\ e^{i(\varphi-\xi)(x)/2}\sin\frac{\theta(x)}{2} & e^{i(\varphi+\xi)(x)/2}\cos\frac{\theta(x)}{2}\end{pmatrix}. \tag{62}$$

We define

$$\Omega_\mu(x)=i\frac{1}{g}U(x)\partial_\mu U^{\dagger}(x). \tag{63}$$

Then, under the gauge transformation, the gauge field $A_\mu(x)$ is transformed to

$$A_\mu(x)\to A'_\mu(x)=U(x)A_\mu(x)U^{\dagger}(x)+\Omega_\mu(x). \tag{64}$$

Ω_μ is evaluated as

$$\begin{aligned}\Omega_\mu &= \frac{1}{g}\Big[\frac{1}{2}(\partial_\mu\varphi\cos\theta)\sigma_3 \\ &+ \left(-\frac{1}{2}\partial_\mu\varphi\cos\xi\sin\theta+\frac{1}{2}\sin\xi\partial_\mu\theta\right)\sigma_1 \\ &+ \left(\frac{1}{2}\partial_\mu\varphi\sin\xi\sin\theta+\frac{1}{2}\cos\xi\partial_\mu\theta\right)\sigma_2\Big]. \end{aligned} \tag{65}$$

When $\xi=-\varphi$, the diagonal term, being proportional to σ_3, is

$$\frac{1}{2g}\partial_\mu\varphi(\cos\theta-1)=-\frac{1}{2g}\left(-\frac{y}{r(r+z)},\frac{x}{r(r+z)},0\right), \tag{66}$$

for $r=\sqrt{x^2+y^2+z^2}$. This term has a singularity $z=-r$ $(\theta=\pi)$ and represents a Dirac monopole. The corresponding gauge field is

$$\mathbf{A}_I=\frac{q_m}{4\pi}\left(-\frac{y}{r(r+z)},\frac{x}{r(r+z)},0\right)=\frac{q_m}{4\pi}\frac{1-\cos\theta}{r\sin\theta}\mathbf{e}_\varphi, \tag{67}$$

where q_m is the magnetic charge and $\mathbf{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$. The magnetic field is

$$\mathbf{B}_m = \nabla \times \mathbf{A}_I = \frac{q_m}{4\pi r^2}\mathbf{e}_r, \tag{68}$$

where $\mathbf{e}_r = \mathbf{r}/r = (x, y, z)/r$. The gauge field which has a singularity at $z = r$ is

$$\mathbf{A}_{II} = \frac{q_m}{4\pi}\left(\frac{y}{r(r-z)}, -\frac{x}{r(r-z)}, 0\right) = -\frac{q_m}{4\pi}\frac{1+\cos\theta}{r\sin\theta}\mathbf{e}_\varphi. \tag{69}$$

This satisfies

$$\nabla \times \mathbf{A}_{II} = \frac{q_m}{4\pi r^2}\mathbf{e}_r. \tag{70}$$

$\mathbf{A}_I$ and $\mathbf{A}_{II}$ are connected by the gauge transformation,

$$\mathbf{A}_I = \mathbf{A}_{II} + \nabla\left(\frac{q_m}{2\pi}\varphi\right). \tag{71}$$

We will see that the gauge field for monopole indeed appears in a multi-band superconductor.

V. HALF-QUANTUM VORTEX

In this section we consider a half-quantum flux vortex in a two-band superconductor[13–15] and discuss an analogy with a monopole. We write the order parameters as

$$\psi_j = \rho_j e^{i\theta_j}, \tag{72}$$

where $\rho_j = |\psi_j|$ is a real quantity. For simplicity, we assume that the coefficients of the Josephson terms are real: $\gamma_{ij} = \gamma_{ji}^* = \gamma_{ji}$. The free energy density is denoted as f, that is, the free energy is given by the integral of f over the space. f is written as

$$\begin{aligned} f &= \sum_j \alpha_j\rho_j^2 + \frac{1}{2}\sum_j \beta_j\rho_j^4 - 2\gamma_{12}\rho_1\rho_2\cos(\theta_1-\theta_2) \\ &- \sum_j K_j\rho_j e^{-i\theta_j}\left(\nabla + i\frac{2\pi}{\phi_0}\mathbf{A}\right)^2(\rho_j e^{i\theta_j}) + \frac{1}{8\pi}\mathbf{H}^2. \end{aligned} \tag{73}$$

We focus on the role of phases of the order parameters; we assume that

$$\rho_j = \rho, \quad K_j = K, \tag{74}$$

and define new phase variables

$$\Phi = \theta_1 + \theta_2, \quad \phi = \theta_1 - \theta_2. \tag{75}$$

The free energy density is

$$\begin{aligned} f &= 2\alpha\rho^2 + \beta\rho^4 - 2\rho^2\gamma_{12}\cos(\phi) - 2K\rho\nabla^2\rho \\ &+ 2K\frac{4\pi^2}{\phi_0^2}\rho^2\mathbf{A}^2 + \frac{1}{2}K\rho^2(\nabla\Phi)^2 \\ &+ K\frac{4\pi}{\phi_0}\rho^2\mathbf{A}\cdot\nabla\Phi + \frac{1}{2}K\rho^2(\nabla\phi)^2 + \frac{1}{8\pi}\mathbf{H}^2 \end{aligned} \tag{76}$$

where $\alpha = (1/2)\sum_j \alpha_j$ and $\beta = (1/2)\sum_j \beta_j$.

Let us focus on the phase difference ϕ. Excitations that are connected to this variable is sometimes called Leggett mode[16]. The equation of motion for ϕ reads

$$2\gamma_{12}\rho^2\sin\phi - K\rho^2\nabla^2\phi = 0. \tag{77}$$

We define $\alpha = 2\gamma_{12}/K$ and adopt that α is positive. The sign of α does not matter because we can change the sign of $\sin\phi$ by shifting the variable ϕ. We consider a one-dimensional-like solution where ϕ has spatial dependence only in one direction, for example, in x direction. In this case the equation for ϕ is

$$\frac{d^2\phi}{dx^2} = \alpha\sin\phi. \tag{78}$$

We use the boundary condition such that $\phi \to 0$ as $x \to -\infty$ and $\phi \to 2\pi$ as $x \to \infty$. Then we have a kink solution:

$$\phi = \pi + 2\sin^{-1}(\tanh(\sqrt{\alpha}x)). \tag{79}$$

The phase difference ϕ changes from 0 to 2π across the kink. This means that θ_1 changes from 0 to π and at the same time θ_2 changes from 0 to $-\pi$. In this case, a half-quantum-flux vortex exists at the end of the kink. This is shown in Fig.2 where the half-quantum vortex is at the edge of the cut (kink). A net change of θ_1 is 2π by a counterclockwise encirclement of the vortex, and that of θ_2 vanishes. Then, we have a half-quantum flux vortex.

The half-quantum vortex can be interpreted as a monopole. Let us assume that there is a cut, namely, kink on the real axis for $x > 0$. The phase θ_1 is represented by

$$\theta_1 = \frac{1}{2}\mathrm{Im}\log\zeta, \tag{80}$$

where

$$\zeta = x + iy. \tag{81}$$

The singularity of θ_j can be transferred to a singularity of the gauge field by a gauge transformation. Let us consider the fictitious z axis perpendicular to the x-y plane. The gauge potential (1-form) is given by

$$A_\pm = -\frac{1}{2}\frac{1}{r(z\pm r)}(ydx - xdy) = \frac{1}{2}(\pm 1 - \cos\theta)d\varphi, \tag{82}$$

where $r = \sqrt{x^2+y^2+z^2}$, and θ and φ are Euler angles. $A_\pm$ correspond to the gauge potential in the upper and lower hemisphere $H_\pm$, respectively. $A_\pm$ are connected by

$$A_+ = A_- + d\varphi. \tag{83}$$

This is the U(1) bundle P over the sphere S^2. At $z = 0$ A_+ coincides with the gauge field A for half-quantum vortex. The Chern class is defined as

$$c_1(P) = -\frac{1}{2}F = -\frac{1}{2}dA_+. \tag{84}$$

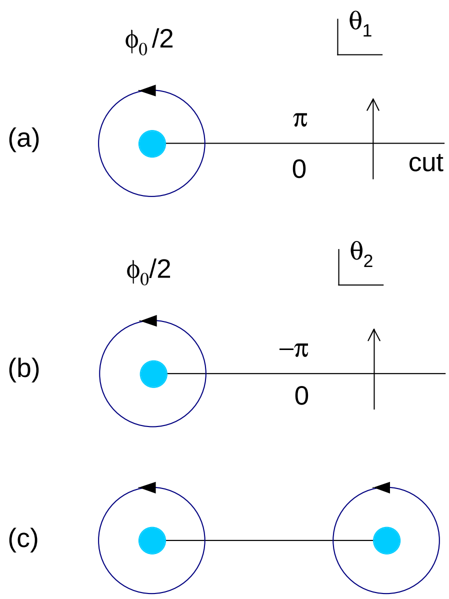

FIG. 2: Half-quantum flux vortex with line singularity. The phase variables θ_1 and θ_2 have line singularities, as shown in (a) and (b). A pair of two half-flux vortices connected by the singularity (domain wall) is shown in (c).

The Chern number is given as

$$\begin{aligned} C_1 &= \int_{S^2} c_1 = -\frac{1}{2\pi}\int_{S^2} F \\ &= -\frac{1}{2\pi}\Big[\int_{H_+} dA_+ \int_{H_-} dA_-\Big] = 1. \end{aligned} \qquad (85)$$

VI. TIME-REVERSAL SYMMETRY BREAKING

Let us consider the case $N \geq 3$. When the gap equation has a complex solution, the time-reversal symmetry is broken. This occurs indeed in a three-band superconductor and leads to time-reversal symmetry breaking. This may be seen as an analogy to the Kobayashi-Maskawa matrix[17]. When two eigenvalues are degenerate, we have a singularity.

The gap equation in eq.(51) is

$$\Delta_i = \sum_j g_{ij} f_j \Delta_j, \qquad (86)$$

where we set

$$f_j = N_j \int d\xi_j \frac{1}{E_j}\tanh\left(\frac{E_j}{2T}\right). \qquad (87)$$

This equation is written as

$$X\Delta = 0, \qquad (88)$$

where $\Delta =^t (\Delta_1, \Delta_2, \cdots)$ and

$$X = G^{-1} - \begin{pmatrix} f_1 & 0 & \cdots \\ 0 & f_2 & \cdots \\ 0 & 0 & \ddots \end{pmatrix}. \qquad (89)$$

The second term in X is a diagonal matrix with elements $(f_1, f_2, \cdots)$. When this equation has a complex solution, the time-reversal symmetry is broken in a superonducting state.

When $N = 3$, the equation $X\Delta = 0$ is written as

$$\begin{pmatrix} \gamma_{11} - f_1 & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} - f_2 & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} - f_3 \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} = 0, \qquad (90)$$

where we defined $\gamma_{ij} = (G^{-1})_{ij}$. We assume that $\gamma_{ij} = \gamma_{ji}$. To obtain the condition that the equation has a complex solution, we adopt that Δ_1 is real and we put $y_j = \mathrm{Im}\Delta_j$. Then we have

$$\begin{pmatrix} \gamma_{12} & \gamma_{13} \\ \gamma_{22} - f_2 & \gamma_{23} \\ \gamma_{32} & \gamma_{33} - f_3 \end{pmatrix} \begin{pmatrix} y_2 \\ y_3 \end{pmatrix} = 0. \qquad (91)$$

This equation has a solution when the rank of the 3×2 matrix equals 1. This means that two column vectors are linearly dependent. This yields

$$\frac{\gamma_{13}}{\gamma_{12}} = \frac{\gamma_{23}}{\gamma_{22} - f_2} = \frac{\gamma_{33} - f_3}{\gamma_{32}}, \qquad (92)$$

where we assume that denominators are nonzero. From the gap equation $\gamma_{21}\Delta_1 + (\gamma_{22} - f_2)\Delta_2 + \gamma_{23}\Delta_3 = 0$, we obtain

$$\frac{\Delta_1}{\gamma_{23}} + \frac{\Delta_2}{\gamma_{31}} + \frac{\Delta_3}{\gamma_{12}} = 0. \qquad (93)$$

In a simple case where three bands are equivalent and $\gamma_{ij}(i \neq j)$ are the same, we have $y_2 = -y_3$ and $\Delta_1 + \Delta_2 + \Delta_3 = 0$. In this case, complex eigenvectors are

$$\begin{pmatrix} 1 \\ e^{i\phi} \\ e^{-i\phi} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ e^{-i\phi} \\ e^{i\phi} \end{pmatrix}, \qquad (94)$$

where $\phi = 2\pi/3$. This state is called the chiral state[19–22]. In this way the time-reversal symmetry broken state is obtained from the gap equation.

VII. FRACTIONAL QUANTUM FLUX VORTICES

In this section let us discuss 1/3-quantized flux vorties in three-band superconductors. The Ginzburg-Landau free energy for a three-band superconductor is

$$\begin{aligned} f &= \sum_j \alpha_j \rho_j^2 + \frac{1}{2}\sum_j \beta_j \rho_j^4 - 2\gamma_{12}\rho_1\rho_2\cos(\theta_1-\theta_2) \\ &- 2\gamma_{23}\rho_2\rho_3\cos(\theta_2-\theta_3) - 2\gamma_{31}\rho_3\rho_1\cos(\theta_3-\theta_1) \\ &- \sum_j K_j\rho_j e^{-i\theta_j}\left(\nabla + i\frac{2\pi}{\phi_0}\mathbf{A}\right)^2(\rho_j e^{i\theta_j}) + \frac{1}{8\pi}\mathbf{H}^2. \end{aligned} \tag{95}$$

Here the order parameters are written as

$$\psi_j = \rho_j e^{i\theta_j}, \tag{96}$$

where $\rho_j = |\psi_j|$ is a real quantity. Since we focus on the role of phases of the order parameters, we assume that

$$\rho_j = \rho, \quad K_j = K, \tag{97}$$

and define new phase variables

$$\Phi = \theta_1+\theta_2+\theta_3, \quad \varphi_1 = \theta_1-\theta_2, \quad \varphi_2 = \theta_2-\theta_3. \tag{98}$$

Then the free energy density is

$$\begin{aligned} f &= 3\alpha\rho^2 + \frac{3}{2}\beta\rho^4 - 2\rho^2[\gamma_{12}\cos(\varphi_1) + \gamma_{23}\cos(\varphi_2) \\ &+ \gamma_{31}\cos(\varphi_1+\varphi_2)] - 3K\rho\nabla^2\rho + 3K\frac{4\pi^2}{\phi_0^2}\rho^2\mathbf{A}^2 \\ &+ \frac{1}{3}K\rho^2(\nabla\Phi)^2 + K\frac{4\pi}{\phi_0}\rho^2\mathbf{A}\cdot\nabla\Phi + \frac{1}{8\pi}\mathbf{H}^2 \\ &+ \frac{1}{3}K\rho^2[(\nabla\varphi_1)^2 + (\nabla\varphi_2)^2 + (\nabla(\varphi_1+\varphi_2))^2], \end{aligned} \tag{99}$$

where $\alpha = (1/3)\sum_j \alpha_j$ and $\beta = (1/3)\sum_j \beta_j$. The stationary conditions with respect to φ_j lead to

$$\gamma_{12}\sin\varphi_1 + \gamma_{31}\sin(\varphi_1+\varphi_2) - \frac{1}{3}K[\nabla^2\varphi_1 + \nabla^2(\varphi_1+\varphi_2)] = 0, \tag{100}$$

$$\gamma_{23}\sin\varphi_1 + \gamma_{31}\sin(\varphi_1+\varphi_2) - \frac{1}{3}K[\nabla^2\varphi_2 + \nabla^2(\varphi_1+\varphi_2)] = 0. \tag{101}$$

Here we examine the ground state of the system with the potential

$$\begin{aligned} V &= -2\gamma_{12}\rho_1\rho_2\cos(\varphi_1) - 2\gamma_{23}\rho_2\rho_3\cos(\varphi_2) \\ &- 2\gamma_{31}\rho_3\rho_1\cos(\varphi_1+\varphi_2). \end{aligned} \tag{102}$$

If we set $\varphi_3 = \theta_3 - \theta_1$, we have $\varphi_1 + \varphi_2 + \varphi_3 = 0$ (mod2π). The minimum of this potential is dependent on the signs of the coefficients $\gamma_{ij}\rho_i\rho_j$ of the Josephson terms. We define $\Gamma_1 = -2\gamma_{12}\rho_1\rho_2$, $\Gamma_2 = -2\gamma_{23}\rho_2\rho_3$, and $\Gamma_3 = -2\gamma_{31}\rho_3\rho_1$. The potential is written as

$$V = \Gamma_1\cos(\varphi_1) + \Gamma_2\cos(\varphi_2) + \Gamma_3\cos(\varphi_3). \tag{103}$$

We assume that the absolute values $|\Gamma_i|$ are almost equal in magnitude. Then there are four cases to be examined as shown in Table I. When all the Γ_i are negative,

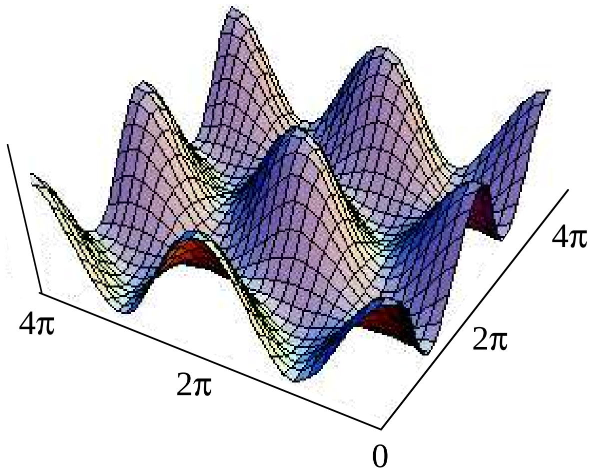

FIG. 3: Contour map of V for $\Gamma_1 = \Gamma_2 = \Gamma_3 > 0$. Black and white dots indicate minima of the potential V. Dotted line is the path in the valley connecting two minima.

TABLE II: Classification of the ground state of the potential V of the Josephson interactions. φ_i in the cases III and IV are for $|\Gamma_1| = |\Gamma_2| = |\Gamma_3|$.

	Γ_1	Γ_2	Γ_3	φ_1	φ_2	φ_3	
Case I	$-$	$-$	$-$	0	0	0	
Case II	$+$	$+$	$-$	π	π	0	
Case III	$-$	$-$	$+$	$\pi/3$	$\pi/3$	$4\pi/3$	chiral state
Case IV	$+$	$+$	$+$	$2\pi/3$	$2\pi/3$	$2\pi/3$	chiral state

we have the minimum at $\varphi_1 = \varphi_2 = \varphi_3 = 0$ (Case I). If we change the sign of Γ_3, this produces a frustration effect and φ_i take fractional values. For example, when all the $|\Gamma_i|$ are equal, we have a minimum at (φ_1,φ_2,φ_3)=($\pi/3$,$\pi/3$,$4\pi/3$). In this state the order parameters are complex and thus the time reversal symmetry is broken. The case IV also exhibits a similar state with fractional values of φ_i. If all the $|\Gamma_i|$ are the same, the ground state is at (φ_1,φ_2,φ_3)=($2\pi/3$,$2\pi/3$,$2\pi/3$) (Fig.3).

In the case with $\gamma_{12} = \gamma_{23}$, we have a solution with $\varphi_1 = \varphi_2 \equiv \varphi$. The variable φ satisfies the double sine-Gordon equation,

$$K\nabla^2\varphi - \gamma_{12}\sin\varphi - \gamma_{31}\sin(2\varphi) = 0. \tag{104}$$

The energy functional is given by

$$E = \int\left[\frac{1}{2}K_0\left(\frac{d\varphi}{dx}\right)^2 + V(\varphi)\right]dx, \tag{105}$$

where $K_0 = 2K\rho^2$ and the potential V is

$$V(\varphi) = V_0\left(\cos\varphi + \frac{u}{2}\cos(2\varphi)\right). \tag{106}$$

We defined $V_0 = -\gamma_{12}\rho^2$ and $u = \gamma_{31}/\gamma_{12}$. Then thre are two cases: (1) $\gamma_{12} < 0$ and (2) $\gamma_{12} > 0$. We show the classification of the double sine-Gordon model in Table II.

In the case $V_0 > 0$ and $u \le 1/2$, there are minima at $\varphi = \pi(\mathrm{mod}2\pi)$ in the potential V. Thus, we have a 2π-kink solution in this case. The boundary conditions should be $\varphi \to \pi$ as $x \to \infty$ and $\varphi \to -\pi$ as $x \to -\infty$, or vice versa. Since we obtain

$$\left(\frac{d\varphi}{dx}\right)^2 = 4\frac{V_0}{K_0}(1-2u)\cos^2\left(\frac{\varphi}{2}\right)\left(1+\frac{2u}{1-2u}\cos^2\left(\frac{\varphi}{2}\right)\right), \tag{107}$$

the kink solution is given by

$$\varphi(x) = \cos^{-1}\left(1 - \frac{2\sinh^2(rx)}{\cosh^2(rx) - 2u}\right), \tag{108}$$

where $r = \sqrt{V_0(1-2u)/K_0}$.

Let us turn the case (2) $V_0 < 0$. There are minima at $\varphi = 0$ (mod2π) for $u > -1/2$, and thus the 2π kink solution exists. We obtain

$$\varphi(x) = \cos^{-1}\left(\frac{2\sinh^2(sx)}{\cosh^2(sx) + 2u} - 1\right), \tag{109}$$

where $s = \sqrt{|V_0|(1+2u)/K_0}$. For large u, the kink shows a characteristic at $x = 0$ because the potential has a local minimum at $\varphi = 0$ for $u > 1/2$. We have a possibility to find some specific features in the excited state due to this anomaly. For $u < -1/2$ we have a fractional-π kink that is given by

$$\varphi(x) = \tan^{-1}\left(\frac{1+2|u|t(x)}{\sqrt{4u^2-1}}\right) - \tan^{-1}\left(\frac{1+2|u|/t(x)}{\sqrt{4u^2-1}}\right), \tag{110}$$

where

$$t(x) = \exp\left(\sqrt{2\frac{|V_0|}{K_0}|u|\left(1-\frac{1}{4u^2}\right)}x\right). \tag{111}$$

For $u = -1$, this chiral solution satisfies the boundary condition that $\varphi \to -\pi/3$ as $x \to -\infty$ and $\varphi \to \pi/3$ as $x \to \infty$.

VIII. FRACTIONAL VORTICES AND BARYONIC BOUND STATES

In general, there are solutions of vortices with fractional quantum flux in multi-band superconductors. Kinks in the space of phase variables θ_j play a central role for the existence of fractional flux vortices as in the case of half-quantum flux vortices.

TABLE III: Classification of the double sine-Gordon model with the potential $V(\varphi) = V_0(\cos(\varphi) + (u/2)\cos(2\varphi))$. $V(\varphi)$ has minima at $\varphi = \varphi_0$ (mod2π).

V_0	u	φ_0	kink
$V_0 > 0$	$u > 1/2$	$\cos^{-1}(-1/(2u))$	fractional-π kink chiral
$V_0 > 0$	$u < 1/2$	π	2π-kink
$V_0 < 0$	$u > -1/2$	0	2π-kink
$V_0 < 0$	$u < -1/2$	$\cos^{-1}(-1/(2u))$	fractional-π kink chiral

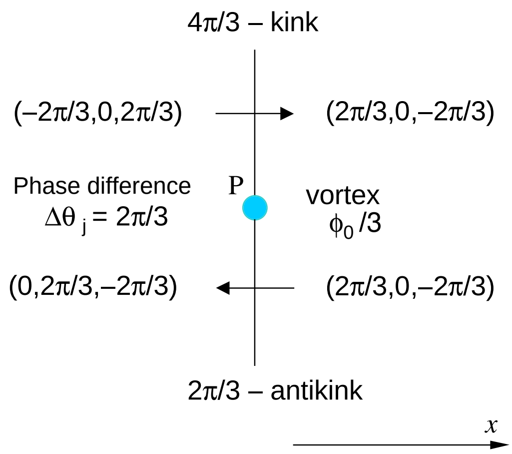

FIG. 4: Kink, antikink and a fractional flux vortex for $\Gamma_1 = \Gamma_2 = \Gamma_3 > 0$. The vortex is at the point P with flux $\phi_0/3$ where ϕ_0 is the flux quantum. We start from $(\theta_1,\theta_2,\theta_3) = (-2\pi/3, 0, 2\pi/3)$ to reach $(0, 2\pi/3, -2\pi/3)$ (modulo 2π) through the $4\pi/3$-kink and $2\pi/3$-antikink. $\varphi_1 = \theta_1 - \theta_2$ goes from $-2\pi/3$ to $2\pi/3$ crossing the $4\pi/3$-kink, and φ_1 goes from $2\pi/3$ to $4\pi/3 \equiv -2\pi/3$ (mod 2π) through the $2\pi/3$-kink.

In three-band superconductors, the fractional-flux vortex exists in the chiral case as well as the non-chiral case. Since we have the fractional-π kink in the chiral state (cases III and IV), the new types of vortices with fractional flux quanta exist on a domain wall of the kink[19–21]. The kink considered in the previous section is a one-dimensional structure in superconductors. There are many types of kinks connecting two minima of the potential in three-band superconductors.

Let us discuss the fractional vortices in the three-band model here. Suppose that two kinks, one is a kink and the other is an antikink, intersect at a point P as shown in Fig.4 in a two-dimensional xy-plane. If a vortex exists along the z axis just at the point P, the vortex should have a fractional flux quantum so that the phase change around the point P is 2π. For $V_0 > 0$ and $u > 1/2$, this is shown schematically in Fig.4. We set the phases

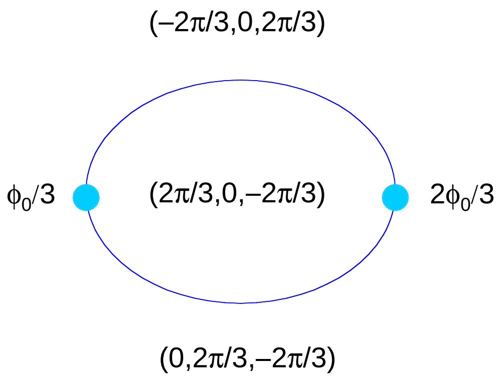

FIG. 5: Two-vortex bound state with line singularities in the time-reversal symmetry broken state. The phase variables θ_i $(i = 1, 2, 3)$ have singularities that are described by kinks. The total flux is ϕ_0. Topologically, the flux $2\phi_0/3$ is equivalent to $-\phi_0/3$. Thus, this state corresponds to the meson under the duality transformation between charge and magnetic flux.

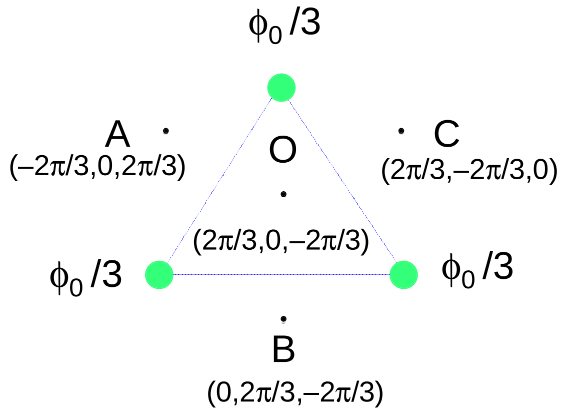

FIG. 6: Three-vortex bound state with line singularities in the time-reversal symmetry broken state. Each vortex has $\phi_0/3$ and the total flux is ϕ_0. The phase variables θ_i $(i = 1, 2, 3)$ have singularities that are fractional-π kinks. One can read one $\phi_0/3$ as $-2\phi_0/3$. Hence, this state corresponds to the neutron.

of the order parameters $(\theta_1, \theta_2, \theta_3) = (-2\pi/3, 0, 2\pi/3)$ in some region. After crossing the $4\pi/3$ kink, they become $(2\pi/3, 0, -2\pi/3)$ where the phase variables φ_1 and φ_2 change from $-2\pi/3$ to $2\pi/3$. If there is also a domain wall of an antikink that starts from the point P as in Fig.4, we have the phases $(0, 2\pi/3, -2\pi/3)$ after we cross the antikink. Here, we obtain the phase difference between the initial and final states (see Fig.4). In this case, the vortex that is located through the point P along the z axis should have a fractional flux quantum $\phi_0/3$. Thus, in the chiral region of three-band superconductors, the existence of fractional vortices is easily concluded in this way.

In the three-band model, the fractional vortex has two line singularities (kinks) in the phases of the gap function as shown in Fig.4. From Fig.4, we have a two-vortex bound state as presented in Fig.5 in the chiral state. Two vortices form a 'molecule' by two kinks. This state may have lower energy than the vortex state with quantum flux ϕ_0 since the magnetic energy $(5/9)\phi_0^2$ is smaller than ϕ_0^2 of the unit flux. The energy of kinks is proportional to the distance R between two fractional vortices if R is large. Thus, the attractive interaction works between them if R is sufficiently large.

Three-vortex bound states are also formulated: they are shown in Figs.6, 7 and 8. The first two figures indicate bound states in the time-reversal symmetry broken state. The last one is for the unbroken state[23]. These states correspond to baryons if we regard the magnetic flux as charge.

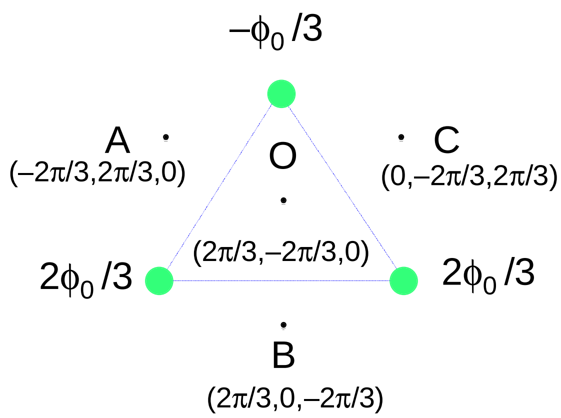

FIG. 7: Three-vortex bound state with line singularities in the time-reversal symmetry broken state. This state corresponds to the proton.

IX. CONFINEMENT AND DECONFINEMENT

In quantum Chromodynamics (QCD), quarks do not appear as asymptotic states. Quarks are constitutes of nucleons and mesons and cannot be separated arbitrary far from the rest of the constituents. This is the hypothesis of confinement[24]. The interquark potential is ex-

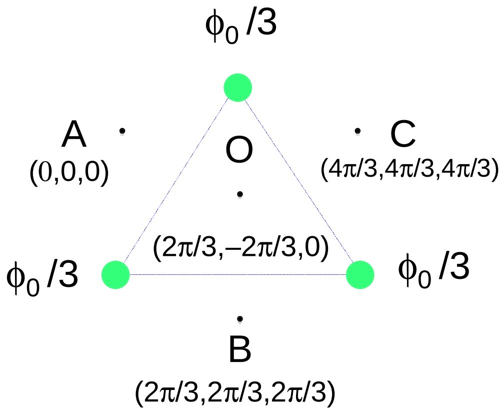

FIG. 8: Three-vortex bound state with line singularities in the time-reversal symmetric state. Each vortex has $\phi_0/3$ and the total flux is ϕ_0. In this state, the region including the point O has higher energy.

pected to be a linearly increasing function so that quarks are confined. To understand such a potential, one may consider a thin flux tube between separated quarks. The energy of the flux tube is proportional to the length of the tube. This is an analogy to magnetic vortices in type-II superconductors where magnetic fields penetrate into superconductors with quantized flux. Because the energy of vortex per unit length is constant, the total energy of vortex is proportional to its length. The confinement of quarks can be understood as confinement in dual superconductor model as a result of the dual to Meissner effect.

Let us consider a pair of half-quantum flux vortices discussed in section V. Because two half-flux vortices are connected by a line of singularity (namely, domain wall), the potential energy between them is proportional to the separation of two vortices,

$$V(r) = \sigma r, \tag{112}$$

for large r where σ is a constant proportional to $\sqrt{K_0 V_0}$. This has an analogy to quarks in a charged pion because of a linear confinement potential. The linear potential in a superconductor originates from the kink in the phase space of the order parameters. The kink of half-quantum flux vortex is a defect in two-dimensional space and this attracts two separated vortices. When two half-quantum vortices become separated far away each other, it is more energetically favorable for a new half-quantum pair to appear, than to let the kink to extend further.

A vortex with unit flux ϕ_0 has the energy proportional to ϕ_0^2, and a half-quantum flux has the magnetic energy which is proportional to $\phi_0^2/4$. Hence, the separation L between two half-flux vortices is determined by the energy balance,

$$\frac{1}{2}\phi_0^2 = AL, \tag{113}$$

for some constant A. Because $A \propto \sqrt{V_0}$ and V_0 is proportional to the strength of Josephson coupling γ, L increases as the Josephson coupling γ decreases.

Goryo et al proposed a mechanism of deconfinement of two vortices due to an entropy effect[25]. They calculated the entropy and estimated the temperature above which two fractional vortices are deconfined. The obtained deconfinement temperature T_{dec} is very close to T_c, and thus free fractional vortices hardly exist at $T < T_c$.

X. SUMMARY

There are a number of interesting analogies between ideas in particle physics and corresponding one in condensed matter physics. Heisenberg considered a microscopic theory of ferromagnets using quantum mechanics, and later employed an analogy with ferromagnets to construct a unified theory of elementary particles[26, 27]. Nambu imported the idea of spontaneous symmetry breaking in condensed-matter physics to particle theory to account for the mass of fermions, especially nucleons. He first notices the resemblance between the Dirac equation in field theory and the gap equation in superconductivity theory.

There are now other analogies between particle physics and physics of superconductivity. A half-quantum flux vortex can be regarded as a monopole, and fractionally quantized-flux vortices bear a striking resemblance to quarks. Two half-quantum vortices form a pair with a linearly increasing potential, which corresponds to mesons. They are not allowed to become separated far away each other for $T < T_c$. In a three-band superconductor, three fractional-flux vortices can form a bound state with the total flux ϕ_0 or 0, which corresponds to the proton or neutron, respectively, after the dual transformation between charge and magnetic flux.

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