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Supersymmetry and the Superconductor-Insulator Transition

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We present a theory of supersymmetric superconductivity and discuss its physical properties. We define the supercharges Q and $Q^\dagger$ satisfying $Q\psi_{\rm BCS} = Q^\dagger\psi_{\rm BCS} = 0$ for the Bardeen-Cooper-Schrieffer state $\psi_{\rm BCS}$. They possess the property expressed by $Q^2 = (Q^\dagger)^2 = 0$, and $\psi_{\rm BCS}$ is the ground state of the supersymmetric Hamiltonian $H = E(QQ^\dagger + Q^\dagger Q)$ for $E > 0$. The superpartners ψ_g and $\psi_{\rm BCS}$ are shown to be degenerate. Here ψ_g denotes a fermionic state within the superconducting gap that exhibits a zero-energy peak in the density of states.

A supersymmetric model of superconductivity with two bands is presented. On the basis of this model we argue that the system of interest goes into a superconducting state from an insulator if an attractive interaction acts between states in the two bands. There are many unusual properties of this model due to an unconventional gap equation stemming from the two-band effect. The model exhibits an unconventional insulator-superconductor first-order phase transition. In the ground state, a first-order transition occurs at the supersymmetric point. We show that certain universal relations in the BCS theory, such as that involving the ratio $\Delta(0)/k_BT_c$, do not hold in the present model.

§1. Introduction

Supersymmetry plays an important role in quantum field theory, quantum mechanics, and condensed-matter physics.[1)–6)] Superconductivity is an important phenomenon that has been studied intensively in condensed matter physics.[7),8)] We believe that supersymmetry also plays a role in superconductivity. Symmetry can sometimes be a key to understanding new phenomena in physics. In recent years, many unconventional superconductors have been reported[9)–13)] and some of them have indicated the coexistence of magnetism and superconductivity.[14)–17)] These results suggest a close relation between superconductivity and magnetism. Novel types of superconductors, such as high-temperature superconductors, are found near the insulating phase. This suggests the possibility of a superconducting instability from an insulator. Thus, it is important to investigate superconductivity near insulators.

Supersymmetry is a symmetry between bosons and fermions. As shown below, the conventional model of superconductivity possesses supersymmetry if we add some terms to the Bardeen-Cooper-Schrieffer (BCS) Hamiltonian. In this supersymmetry, the superpartner of the Cooper pair (boson) is a fermionic state in the superconducting gap. This fermionic state describes a bound state in the gap, which, in some cases, has magnetism coexisting with superconductivity. The $SO(5)$ theory[18)] is an attempt to unify superconductivity and magnetism as a representation of the symmetry group $SO(5)$. We propose the idea that the paired state and fermionic excitation can be regarded as superpartners.

In this paper, we construct a supersymmetric Hamiltonian which describes superconductivity, and discuss its physical properties. We define Q and $Q^\dagger$ so that the BCS state is an eigenstate of the supersymmetric Hamiltonian $H = E(QQ^\dagger + Q^\dagger Q)$. Further, the BCS state is shown to be supersymmetric invariant, i.e., that it satisfies the relation

$$Q\psi_{\mathrm{BCS}} = Q^\dagger \psi_{\mathrm{BCS}} = 0. \tag{1·1}$$

The fermionic state in the gap exhibits a peak in the density of states within the gap.

In a supersymmetric theory of superconductivity, there are many unusual properties stemming from an unconventional gap equation. We present a supersymmetric two-band model with an energy gap between two bands. This system goes into a superconducting phase from an insulator if an attractive interaction acts between states in the two bands. We show that this is an unconventional insulator-superconductor first-order phase transition.

This paper is organized as follows. In §2 the algebra for superconductivity is examined. In §3 a supersymmetric Hamiltonian for superconductivity is presented. In §4 the density of states is calculated, and we give an investigation of the electron tunneling through a normal metal-superconductor junction. In §5 supersymmetry in a two-band system is investigated. We show that there is a first-order transition from a superconductor to an insulator if we vary the hybridization matrix between the two bands. We give a summary in the last section.

§2. Supersymmetric quantum mechanics and algebra for superconductivity

Our theory is based on a supersymmetry algebra for fermions and bosons. Supersymmetric quantum mechanics is described by the Hamiltonian

$$H = E(QQ^\dagger + Q^\dagger Q) \tag{2·1}$$

for supercharges Q and $Q^\dagger$ and $E > 0$. The supercharges Q and $Q^\dagger$ transform the bosonic state to the corresponding fermionic state, and vice versa. The simplest form of supersymmetric quantum mechanics is given by generators, $Q = \psi^\dagger b$ and $Q^\dagger = b^\dagger \psi$, for fermions ψ and bosons b. If we assume $[b, \psi] = [b, \psi^\dagger] = 0$, the Hamiltonian is given by $H = E(QQ^\dagger + Q^\dagger Q) = E(b^\dagger b + \psi^\dagger \psi)$ $(E > 0)$. If we choose b to be the operator of the harmonic oscillator, $b = (ip+x)/\sqrt{2}$ and $b^\dagger = (-ip+x)/\sqrt{2}$, the Hamiltonian is the supersymmetric harmonic oscillator, given by

$$H = E(p^2/2 + x^2/2 + [\psi^\dagger, \psi]/2). \tag{2·2}$$

The ground state is the lowest energy state of the harmonic oscillator with no fermions. An extension of the harmonic oscillator can be straightforwardly obtained by introducing a superpotential $W = W(x)$[19] as $b = (ip + dW/dx)/\sqrt{2}$

and $b^\dagger = (-ip + dW/dx)/\sqrt{2}$. If we assume $dW/dx = \lambda x$, the Hamiltonian is

$$H = E(b^\dagger b + \lambda \psi^\dagger \psi). \tag{2·3}$$

The square root of the superconducting Hamiltonian is first necessary to construct a supersymmetric model of superconductivity. For this purpose, we extend simple supersymmetric quantum mechanics to a system with two fermions, represented by ψ_1 and ψ_2, and a boson, represented by b. If ψ_i and b obey the fermionic and bosonic commutation relations, $\{\psi_i, \psi_j^\dagger\} = \delta_{ij}$, $[b, b^\dagger] = 1$, and $[\psi_i, b] = [\psi_i, b^\dagger] = 0$, an extension is trivial. In order to examine a non-trivial quantum system with two fermions, we consider the algebra characterized by the following commutation relations for the fermions ψ_1 and ψ_2 and the boson b:

$$\{\psi_i, \psi_i^\dagger\} = 1, \quad (i = 1, 2) \tag{2·4}$$

$$\{\psi_1, \psi_2\} = \{\psi_1, \psi_2^\dagger\} = 0, \tag{2·5}$$

$$[\psi_1^\dagger, b] = \psi_2, \tag{2·6}$$

$$[\psi_2^\dagger, b] = -\psi_1, \tag{2·7}$$

$$[\psi_1, b] = 0, \tag{2·8}$$

$$[\psi_2, b] = 0, \tag{2·9}$$

$$[b, b^\dagger] = 1 - \psi_1^\dagger \psi_1 - \psi_2^\dagger \psi_2. \tag{2·10}$$

This algebra contains the commutation relations for Cooper pairs and fermions with spin up and spin down. We impose the condition of $b^2 = 0$, since b is the operator for the Cooper pair. The relation $b^2 = 0$ implies $[b^2, \psi_i] = 0$ $(i = 1, 2)$, which leads to

$$\psi_1 b = b\psi_1 = \psi_2 b = b\psi_2 = 0. \tag{2·11}$$

We refer to this set of commutation relations as the BCS algebra in this paper. Supercharges are defined as

$$Q = v^* b \psi_1^\dagger + u b^\dagger \psi_2, \tag{2·12}$$

$$Q^\dagger = v \psi_1 b^\dagger + u \psi_2^\dagger b, \tag{2·13}$$

where u (which is real) and v are constants satisfying $u^2 + |v|^2 = 1$. It is easy to show the nilpotency of Q and $Q^\dagger$ employing the above algebraic relations. The Hamiltonian is then defined by

$$H = 2E(QQ^\dagger + Q^\dagger Q) \tag{2·14}$$

for a constant $E > 0$. The factor 2 is included for later convenience. The bosonic states are given by linear combinations of $|0\rangle$ and $b^\dagger|0\rangle$. The matrix elements of H for these basis states are

$$\begin{pmatrix} |v|^2 & -uv^* \\ -uv & u^2 \end{pmatrix}. \tag{2·15}$$

Then, the eigenstates are given by the BCS state $\psi_{\mathrm{BCS}} = (u + vb^\dagger)|0\rangle$ and $\psi_{\mathrm{BCS}}^\perp = (v^* - ub^\dagger)|0\rangle$, which is orthogonal to ψ_{BCS}. Here, $|0\rangle$ denotes the vacuum: $b|0\rangle =$

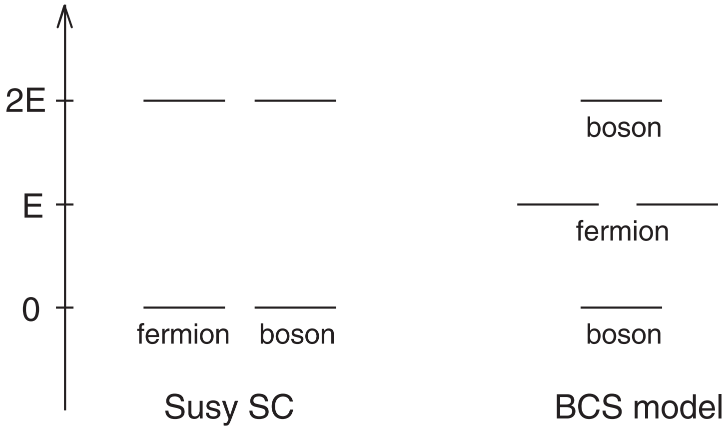

Fig. 1. Energy levels of the supersymmetric superconductivity models.

$\psi_i|0\rangle = 0$. The fermionic states are $\psi_g = \psi_1^\dagger|0\rangle$ and $\psi_e = \psi_2^\dagger|0\rangle$. We can show that Q and $Q^\dagger$ annihilate both $\psi_{\rm BCS}$ and ψ_g:

$$Q\psi_{\rm BCS} = Q^\dagger\psi_{\rm BCS} = 0, \tag{2·16}$$

$$Q\psi_g = Q^\dagger\psi_g = 0. \tag{2·17}$$

Thus, $\psi_{\rm BCS}$ and ψ_g are supersymmetric ground states. $\psi_{\rm BCS}^\perp$ and ψ_e have the eigenvalue $2E$ and are superpartners; i.e., they are transformed to each other by Q and $Q^\dagger$:

$$Q\psi_e = -\psi_{\rm BCS}^\perp, \quad Q^\dagger\psi_{\rm BCS}^\perp = -\psi_e. \tag{2·18}$$

In this model, fermionic and bosonic states are always degenerate. We present the energy scheme in Fig. 1, and the energy levels for the BCS model are also displayed for comparison. In the BCS model, the fermionic excited states have the energy E.

§3. Supersymmetric Hamiltonian

There are several ways to express fermions ψ_1 and ψ_2 in terms of the conduction electrons with wave number $\boldsymbol{k}$. If we write $\psi_1(k) = c_{k\uparrow}$, $\psi_2(k) = -c_{-k\downarrow}$, and $b_k = c_{-k\downarrow}c_{k\uparrow}$ for each wave number $\boldsymbol{k}$, the supersymmetric charges Q_k and $Q_k^\dagger$ are given by

$$Q_k = v_k^* b_k c_{k\uparrow}^\dagger - u_k b_k^\dagger c_{k\downarrow} = v_k^* c_{-k\downarrow}(1-n_{k\uparrow}) - u_k c_{k\uparrow}^\dagger n_{-k\downarrow}, \tag{3·1}$$

$$Q_k^\dagger = v_k c_{k\uparrow} b_k^\dagger - u_k c_{-k\downarrow}^\dagger b_k = v_k(1-n_{k\uparrow})c_{-k\downarrow}^\dagger - u_k n_{-k\downarrow}c_{k\uparrow}. \tag{3·2}$$

Then, the Hamiltonian is given by

$$\begin{aligned} H &= \sum_k 2E_k(Q_kQ_k^\dagger + Q_k^\dagger Q_k) \\ &= \sum_k 2E_k|v_k|^2 + \sum_k\{\xi_k(c_{k\uparrow}^\dagger c_{k\uparrow} + c_{-k\downarrow}^\dagger c_{-k\downarrow}) \end{aligned}$$

$$-E_k(c_{k\uparrow}^\dagger c_{k\uparrow} - c_{-k\downarrow}^\dagger c_{-k\downarrow}) - (\Delta_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + \Delta_k^* c_{-k\downarrow} c_{k\uparrow})\}, \tag{3·3}$$

where $\xi_k/E_k = u_k^2 - |v_k|^2$ and $\Delta_k/E_k = 2u_k v_k$. We set $\xi_k = \epsilon_k - \mu$, where ϵ_k is the electron dispersion relation and μ is the chemical potential. The superconducting gap Δ_k should be determined self-consistently. The BCS state,

$$\psi_{\rm BCS} = \prod_k (u_k + v_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger)|0\rangle, \tag{3·4}$$

is the ground state of H as we have

$$Q_k \psi_{\rm BCS} = Q_k^\dagger \psi_{\rm BCS} = 0. \tag{3·5}$$

The fermionic state $\psi_g = c_{k\uparrow}^\dagger|0\rangle$ constructed from ψ_1 is also the supersymmetric ground state. The third term on the right-hand side of Eq.(3·3) is missing in the original BCS Hamiltonian, and thus the degeneracy is lifted in the BCS theory. In the BCS model, the fermionic excited state has energy E_k, while in the present model, one fermion state is degenerate with the BCS state and the other fermion state has energy $2E_k$. The operators Q_k and $Q_k^\dagger$ resemble the Bogoliubov operators $\alpha_{k\sigma}$, which annihilate the BCS state as $\alpha_{k\sigma}\psi_{\rm BCS} = 0$. Note that $\alpha_{k\sigma}^\dagger$ creates the fermionic excited state $\alpha_{k\sigma}^\dagger \psi_{\rm BCS}$ with eigenvalue E_k.

In general, we can rotate $(\psi_1(k), \psi_2(k))$ in the space spanned by $(c_{k\uparrow}, -c_{-k\downarrow})$:

$$\begin{pmatrix} \psi_1(k) \\ \psi_2(k) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ -c_{-k\downarrow} \end{pmatrix}, \tag{3·6}$$

and $b_k = c_{-k\downarrow}c_{k\uparrow} = \psi_1(k)\psi_2(k)$. The same commutators are derived for ψ_1, ψ_2 and b_k. Then, the Hamiltonian reads

$$\begin{aligned} H = \sum_k 2E_k|v_k|^2 + \sum_k \{ &\xi_k(c_{k\uparrow}^\dagger c_{k\uparrow} + c_{-k\downarrow}^\dagger c_{-k\downarrow}) \\ &- E_k\Big[\cos(2\theta)(n_{k\uparrow} - n_{-k\downarrow}) + \sin(2\theta)(c_{k\uparrow}^\dagger c_{-k\downarrow} + c_{-k\downarrow}^\dagger c_{k\uparrow})\Big] \\ &- \Delta_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger - \Delta_k^* c_{-k\downarrow}c_{k\uparrow}\}. \end{aligned} \tag{3·7}$$

The second term corresponds to rotation by an angle 2θ multiplied by the matrix diag$(1,-1)$:

$$\begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3·8}$$

§4. Density of states and electron tunneling

Now let us examine the physical properties of our model. We investigate the following Hamiltonian for this purpose:

$$\begin{aligned} H_a = \sum_k 2E_k|v_k|^2 + \sum_k \{ &\xi_k(c_{k\uparrow}^\dagger c_{k\uparrow} + c_{-k\downarrow}^\dagger c_{-k\downarrow}) \\ &- h_k(c_{k\uparrow}^\dagger c_{k\uparrow} - c_{-k\downarrow}^\dagger c_{-k\downarrow}) - (\Delta_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + \Delta_k^* c_{-k\downarrow}c_{k\uparrow})\}. \end{aligned} \tag{4·1}$$

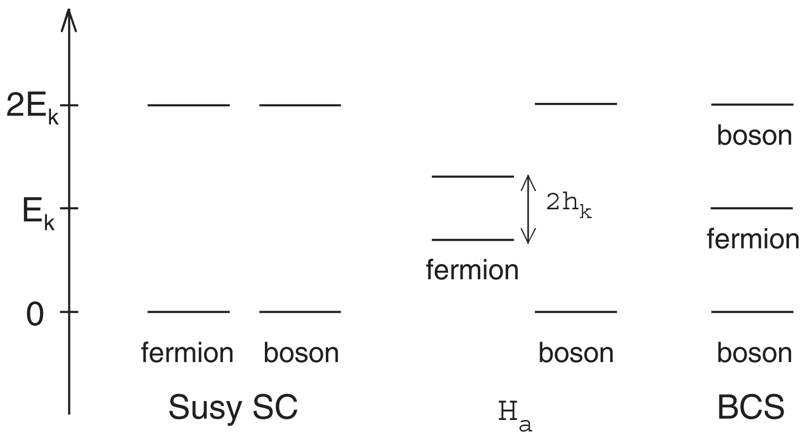

Fig. 2. Energy levels of superconductivity models for each k.

This Hamiltonian reduces to that of the BCS model for $h_k = 0$ and to the supersymmetric one for $h_k = E_k$. The level structure of the Hamiltonian H_a is displayed in Fig. 2, and it is seen that it connects the BCS model to the supersymmetric superconductivity model. We define the Green functions as

$$G_{\sigma\sigma'}(\tau, \boldsymbol{k}) = -\langle T c_{k\sigma}(\tau) c^{\dagger}_{k\sigma'}(0)\rangle, \tag{4·2}$$

$$F_{-\sigma\sigma'}(\tau, \boldsymbol{k}) = \langle T c_{-k-\sigma}(\tau) c_{k\sigma'}(0)\rangle, \tag{4·3}$$

$$F^{+}_{-\sigma\sigma'}(\tau, \boldsymbol{k}) = \langle T c^{\dagger}_{-k-\sigma}(\tau) c^{\dagger}_{k\sigma'}(0)\rangle. \tag{4·4}$$

The Fourier transforms are

$$G_{\sigma\sigma'}(\tau, \boldsymbol{k}) = \frac{1}{\beta}\sum_n e^{-i\omega_n\tau} G_{\sigma\sigma'}(i\omega_n, \boldsymbol{k}), \tag{4·5}$$

$$F^{+}_{-\sigma\sigma'}(\tau, \boldsymbol{k}) = \frac{1}{\beta}\sum_n e^{-i\omega_n\tau} F^{+}_{-\sigma\sigma'}(i\omega_n, \boldsymbol{k}), \tag{4·6}$$

where $\omega_n = (2n+1)\pi/\beta$ ($\beta = 1/(k_B T)$). From the equations of motion for the Green functions, we obtain

$$G_{\sigma\sigma'}(i\omega_n, \boldsymbol{k}) = \delta_{\sigma\sigma'} \frac{i\omega_n + \xi_k + \sigma h_k}{(i\omega_n - \xi_k + \sigma h_k)(i\omega_n + \xi_k + \sigma h_k) - |\Delta_k|^2}, \tag{4·7}$$

$$F^{+}_{-\sigma\sigma'}(i\omega_n, \boldsymbol{k}) = \delta_{\sigma\sigma'} \frac{\sigma \Delta^{*}_k}{(i\omega_n - \xi_k + \sigma h_k)(i\omega_n + \xi_k + \sigma h_k) - |\Delta_k|^2}, \tag{4·8}$$

where we assume that $\xi_{-k} = \xi_k$ and $h_{-k} = h_k$. We assume the isotropic gap function $\Delta_k = \Delta$. Then, the density of states for $h_k = E_k$ is given by

$$\rho(\omega) = -\frac{1}{\pi}\frac{1}{V}\sum_{k\sigma} \mathrm{Im} G_{\sigma\sigma}(\omega + i\delta, \boldsymbol{k}), \tag{4·9}$$

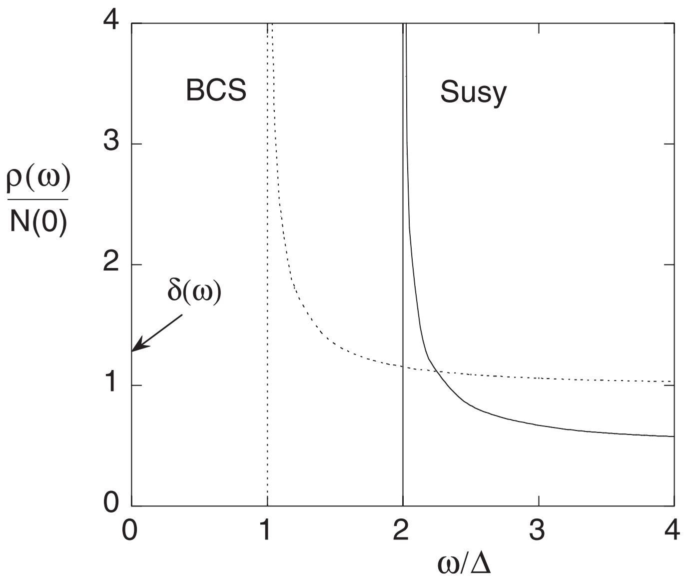

Fig. 3. Density of states for the supersymmetric (susy) superconductivity model. The dotted lines denote those for the BCS model.

where V is the volume of the system. This function has peaks at $\omega = 0$ and $\omega = 2\Delta$, as shown in Fig. 3:

$$\rho(\omega) = \delta(\omega) + N(0)\frac{1}{2}\frac{|\omega|}{\sqrt{\omega^2 - (2\Delta)^2}}. \tag{4·10}$$

If we set $h_k = \alpha E_k$ $(0 \le \alpha \le 1)$, we have peaks at $\omega = (1-\alpha)\Delta$ and $(1+\alpha)\Delta$. The lower peak becomes the zero-energy peak at the supersymmetric point $\alpha = 1$. In other words, the zero-energy peak splits into two peaks as the supersymmetry is broken.

Because the supersymmetric model has a zero-energy peak, we expect anomalous behavior for transport properties. To elucidate this point, we investigate electron tunneling through the normal metal-superconductor junction in this section for the supersymmetric case. The current I is given by[20]

$$I = 2e\sum_{kp}|T_{kp}|^2\int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi}A_R(\boldsymbol{k}.\epsilon)A_L(\boldsymbol{p},\epsilon + eV_b)(f(\epsilon) - f(\epsilon + eV_b)), \tag{4·11}$$

for bias voltage V_b, where T_{kp} is the transition coefficient of the junction, and $f(\epsilon)$ is the Fermi distribution function, $f(\epsilon) = 1/(e^{\beta\epsilon} + 1)$. The quantities A_L and A_R are spectral functions for a normal metal and superconductor, respectively, defined as $A(\boldsymbol{p},\omega) = -\sum_\sigma \mathrm{Im}G_{\sigma\sigma}(\omega + i\delta, \boldsymbol{p})$ with the retarded Green function. Because $A_L(\boldsymbol{p},\epsilon) = 2\pi\delta(\epsilon - \xi_p)$ and

$$A_R(\boldsymbol{k},\epsilon) = \pi(\delta(\epsilon) + u_k^2\delta(\epsilon - 2E_k) + v_k^2\delta(\epsilon + 2E_k)), \tag{4·12}$$

for the supersymmetric Hamiltonian, the current I is

$$
\begin{aligned}
I &= 2e\pi \sum_{kp} |T_{kp}|^2 \int_{-\infty}^{\infty} d\epsilon[\delta(\epsilon) + u_k^2\delta(\epsilon - 2E_k) \\
&\quad +v_k^2\delta(\epsilon + 2E_k)]\delta(\epsilon + eV_b - \xi_p)(f(\epsilon) - f(\epsilon + eV_b)) \\
&= 2e\pi \sum_{kp} |T_{kp}|^2 \left[u_k^2\delta(eV_b + 2E_k - \xi_p)(f(2E_k) - f(\xi_p))\right. \\
&\quad +v_k^2\delta(eV_b - 2E_k - \xi_p)(f(-2E_k) - f(\xi_p)) \\
&\quad \left.+\delta(eV_b - \xi_p)(f(0) - f(\xi_p))\right].
\end{aligned} \tag{4·13}
$$

At the zero temperature, we have

$$
\begin{aligned}
I &= 2e\pi N_R(0)N_L(0)\, |T|^2 \int_{-\infty}^{\infty} d\xi_p \int_{-\infty}^{\infty} d\xi_k \\
&\quad \times[-u_k^2 f(\xi_p)\delta(eV_b + 2E_k - \xi_p) \\
&\quad +v_k^2(1 - f(\xi_p))\delta(eV_b - 2E_k - \xi_p) \\
&\quad +\delta(eV_b - \xi_p)(f(0) - f(\xi_p))] \\
&= 2e\pi N_R(0)N_L(0)\, |T|^2 \int_{-\infty}^{\infty} d\xi_k[-u_k^2 f(eV_b + 2E_k) \\
&\quad +v_k^2(1 - f(eV_b - 2E_k)) + f(0) - f(eV_b)],
\end{aligned} \tag{4·14}
$$

where $|T_{kp}|^2$ is approximated as $|T|^2$. Then for $eV_b \geq 0$, we obtain

$$
\begin{aligned}
I &= 2e\pi N_R(0)N_L(0)\, |T|^2 \sqrt{(eV_b/2)^2 - \Delta^2}\,\theta\left(\frac{eV_b}{2} - \Delta\right) \\
&\quad +\pi N_L(0)\, |T|^2 \,(f(0) - f(eV_b)).
\end{aligned} \tag{4·15}
$$

The differential conductance is evaluated as

$$
\begin{aligned}
\frac{dI}{d(eV_b)} &= 2e\pi N_R(0)N_L(0)\, |T|^2 \frac{eV_b}{\sqrt{(eV_b)^2 - (2\Delta)^2}}\theta\left(\frac{eV_b}{2} - \Delta\right) \\
&\quad +\pi N_L(0)\, |T|^2 \left(-\frac{\partial f(eV_b)}{\partial(eV_b)}\right).
\end{aligned} \tag{4·16}
$$

The second term, which results from the supersymmetric effect, leads to a peak at $eV_b = 0$. Supersymmetric superconductivity may provide a model for the zero-bias peak at the junction of unconventional superconductors.[21)]

§5. Insulator-superconductor transition – a two-band model

Let us start with a two-band system in order to study the model with supersymmetry. We consider the Hamiltonian

$$
H_{2\text{-band}} = \sum_k [\xi_k^a a_k^\dagger a_k + \xi_k^b b_k^\dagger b_k + v(a_k^\dagger b_k + b_k^\dagger a_k)], \tag{5·1}
$$

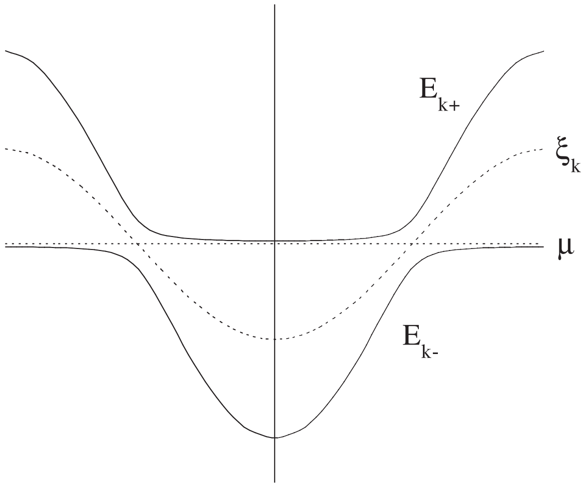

Fig. 4. Dispersion relation of the two-band model as a function of the wave number.

where a_k and b_k are fermion operators. This Hamiltonian can be written as

$$H_{2\text{-band}} = \sum_k (E_k^- \alpha_k^\dagger \alpha_k + E_k^+ \beta_k^\dagger \beta_k), \tag{5·2}$$

where α_k and β_k are linear combinations of a_k and b_k, and

$$E_k^\pm = (\xi_k^a + \xi_k^b)/2 \pm \sqrt{(\xi_k^a - \xi_k^b)^2/4 + v^2}. \tag{5·3}$$

For the localized band $\xi_b = 0$ (at the level of the chemical potential), we have the dispersion relation

$$E_k^\pm = \xi_k \pm \sqrt{\xi_k^2 + v^2}, \tag{5·4}$$

where $\xi_k = \xi_k^a/2$. Here we assume that $\xi_{-k} = \xi_k$. The band structure is shown in Fig. 4. The Fermi level is in the gap, and thus the system is insulating in the normal state. Let us consider the Hamiltonian with the pairing term:

$$\begin{aligned} H = &\sum_k [\xi_k(\alpha_k^\dagger \alpha_k + \beta_k^\dagger \beta_k) - \sqrt{\xi_k^2 + v^2}(\alpha_k^\dagger \alpha_k - \beta_k^\dagger \beta_k)] \\ &- \sum_k (\Delta \alpha_k^\dagger \beta_{-k}^\dagger + \Delta^* \beta_{-k} \alpha_k). \end{aligned} \tag{5·5}$$

If $v = \Delta$, this Hamiltonian has exact supersymmetry. In the following we investigate the properties of this model near the supersymmetric point, regarding v as a parameter.

Let us consider the Hamiltonian

$$\begin{aligned} H_g = &\sum_k \left[\xi_k(\alpha_k^\dagger \alpha_k + \beta_k^\dagger \beta_k) - \sqrt{\xi_k^2 + v^2}(\alpha_k^\dagger \alpha_k - \beta_k^\dagger \beta_k)\right] \\ &+ \frac{g}{V} \sum_{kk'q} \alpha\dagger_{k+q} \beta_{k'-q}^\dagger \beta_{k'} \alpha_k, \end{aligned} \tag{5·6}$$

where we assume $g < 0$ and ignore the $\boldsymbol{k}$-dependence of g for simplicity. The third term represents the attractive interaction. A similar two-band model was investigated in Ref. 22). Using the mean-field theory we obtain the Hamiltonian in Eq. (5·5) for Δ defined as

$$\Delta = \frac{g}{V}\sum_{k'}\langle \alpha_{k'}\beta_{-k'}\rangle. \tag{5·7}$$

In the supersymmetric case, $v = \Delta$, the paired state $(u_k + v_k\alpha_k^\dagger\beta_{-k}^\dagger)|0\rangle$ and the unpaired fermionic state are degenerate. If v is large, i.e. if $v > \Delta$, the superconducting state is unstable, and the ground state is a band insulator with an occupied lower band. Thus, there is a first-order transition at $v = \Delta$ from a superconductor to an insulator.

We define the following Green functions:

$$G_\alpha(\tau, \boldsymbol{k}) = -\langle T\alpha_k(\tau)\alpha_k^\dagger(0)\rangle, \tag{5·8}$$

$$F_{\beta\alpha}^+(\tau, \boldsymbol{k}) = \langle T\beta_{-k}^\dagger(\tau)\alpha_k^\dagger(0)\rangle. \tag{5·9}$$

Their Fourier transforms are defined similarly to those in Eq. (4·5). The equations of motion read

$$(i\omega_n - E_k^-)G_\alpha(i\omega_n, \boldsymbol{k}) - \Delta F_{\beta\alpha}^+(i\omega_n, \boldsymbol{k}) = 1, \tag{5·10}$$

$$(i\omega_n + E_k^+)F_{\beta\alpha}^+(i\omega_n, \boldsymbol{k}) - \Delta^* G_\alpha(i\omega_n, \boldsymbol{k}) = 0, \tag{5·11}$$

Thus we have

$$F_{\beta\alpha}^+(i\omega_n, k) = \frac{\Delta^*}{(i\omega_n - E_k^-)(i\omega_n + E_k^+) - |\Delta|^2}. \tag{5·12}$$

The gap equation is

$$\begin{aligned} 1 &= g\frac{1}{V}\sum_k \frac{1}{\beta}\sum_n \frac{1}{(i\omega_n)^2 + 2\sqrt{\xi_k^2 + v^2}\, i\omega_n - (|\Delta|^2 - v^2)} \\ &= |g|\frac{1}{V}\sum_k \frac{1}{2\sqrt{\xi_k^2 + |\Delta|^2}}\Big(1 - f\Big(\sqrt{\xi_k^2 + |\Delta|^2} + \sqrt{\xi_k^2 + v^2}\Big) \\ &\quad - f\Big(\sqrt{\xi_k^2 + |\Delta|^2} - \sqrt{\xi_k^2 + v^2}\Big)\Big), \end{aligned} \tag{5·13}$$

where V is the volume of the system. At the zero temperature, $T = 0$, we have a solution if we assume that $\Delta(T = 0) > v$:

$$\Delta_0 = 2\omega_0 \mathrm{exp}(-1/(|g|N(0))), \tag{5·14}$$

where $N(0)$ is the density of states at the Fermi level and ω_0 is the cutoff energy. Here, $\Delta(T = 0)$ is a step function as a function of v:

$$\begin{aligned} \Delta(T = 0) &= \Delta_0 \text{ if } v < \Delta_0, \\ &= 0 \quad \text{if } v > \Delta_0. \end{aligned} \tag{5·15}$$

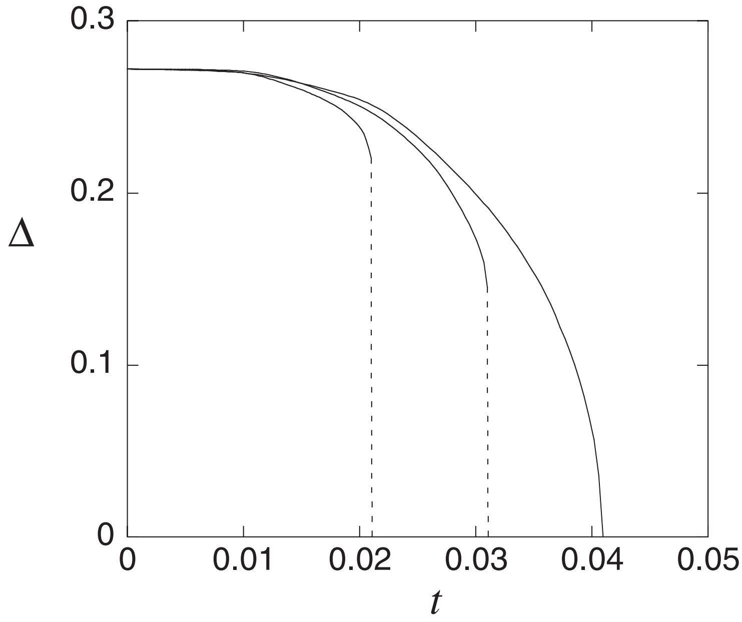

Fig. 5. Superconducting gap as a function of the temperature $t = k_BT/\omega_0$ for $v/\omega_0 = 0$, 0.05 and 0.1 (from the top). Here we set $\lambda = 1/2$.

A finite strength of the coupling constant $|g|N(0)$, with the condition $v < \Delta_0$, is needed to produce superconductivity. This is because the transition is from the insulating state without the Fermi surface. In the ground state, there occurs a first-order transition at the supersymmetric point $v = \Delta_0$ from a superconductor to an insulator if we vary the parameter v. We define the dimensionless coupling constant λ as $\lambda = |g|N(0)$. The function, $\Delta(T)$, obtained numerically, is shown in Fig. 5 as a function of the temperature for $v = 0$, 0.05 and 0.1 and $\lambda = 1/2$. A first-order transition occurs for $v = 0.05$ and 0.1 as seen in Fig. 5. The transition is first order at finite T, except in the region of small v, where the transition is second order. The critical temperature $t_c = k_BT_c/\omega_0$ is a decreasing function of v, as is shown in Fig. 6, and it vanishes for $v > \Delta_0$.

A superconductor-insulator transition occurs at $T = T_c$. The gap equation in Eq. (5·13) is written

$$\frac{1}{\lambda} = \int_0^{\omega_0} d\xi \frac{1}{\sqrt{\xi^2+\Delta^2}}(1 - f(\sqrt{\xi^2+\Delta^2}+\sqrt{\xi^2+v^2}) \\ -f(\sqrt{\xi^2+\Delta^2}-\sqrt{\xi^2+v^2})), \tag{5·16}$$

where we set $|\Delta| = \Delta$. The right-hand side of this equation has a maximum for $T > 0$ at low temperatures, while it is a decreasing function at high temperatures (see Fig. 7). There is no solution if the maximum is less than $1/\lambda$, and there are two solutions if $1/\lambda$ is less than the maximum. The first-order transition is realized if $1/\lambda$ is equal to the maximum. The larger gap is shown in Fig. 5 because it is connected to the gap at $T = 0$. It is important to note that the ratio $2\Delta/(k_BT_c)$ is

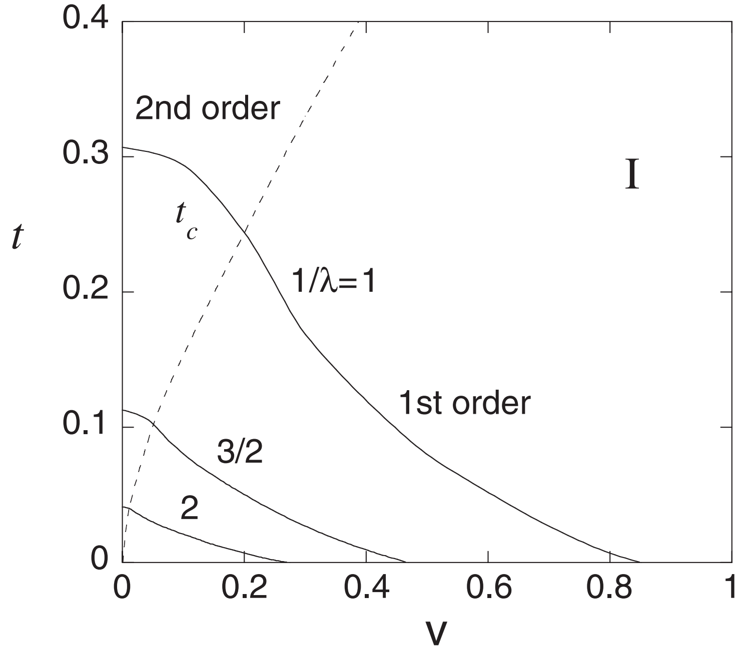

Fig. 6. Critical temperature $t_c = k_B T_c/\omega_0$ as a function of v/ω_0 for $1/\lambda = 1$, $3/2$ and 2 (from the top). Here ω_0 is taken as the unit of energy. The transition is first order on the left-hand side of the dashed line, and second order on the other side. The insulating phase exists above t_c.

larger than the BCS value, 3.53. In the limit $v \to 0$, the gap equation for T_c becomes

$$1 = |g|\frac{1}{V}\sum_k \frac{1/2 - f(2|\xi_k|)}{2|\xi_k|}, \tag{5·17}$$

from which we obtain

$$k_B T_c(v = 0) = \frac{2e^\gamma}{\pi} 2\omega_0 \exp(-2/(|g|N(0))). \tag{5·18}$$

Then the ratio at $T = 0$,

$$\frac{2\Delta_0}{k_B T_c(v = 0)} = \frac{\pi}{e^\gamma} e^{1/\lambda} \tag{5·19}$$

is much larger than $2\pi/e^\gamma = 3.53$ where $\gamma = 0.5772$ the Euler constant. Figure 8 plots this ratio as a function of v. We see that it diverges at the supersymmetric point $v = \Delta_0$. Thus $\Delta(0)/k_B T_c$ does not follow the universal relation of the BCS theory.

§6. Discussion

We have shown that the BCS state is invariant under the supersymmetric transformation generated by Q and $Q^\dagger$. The BCS state is the ground state of the supersymmetric Hamiltonian. The superpartner is also the supersymmetric ground state,

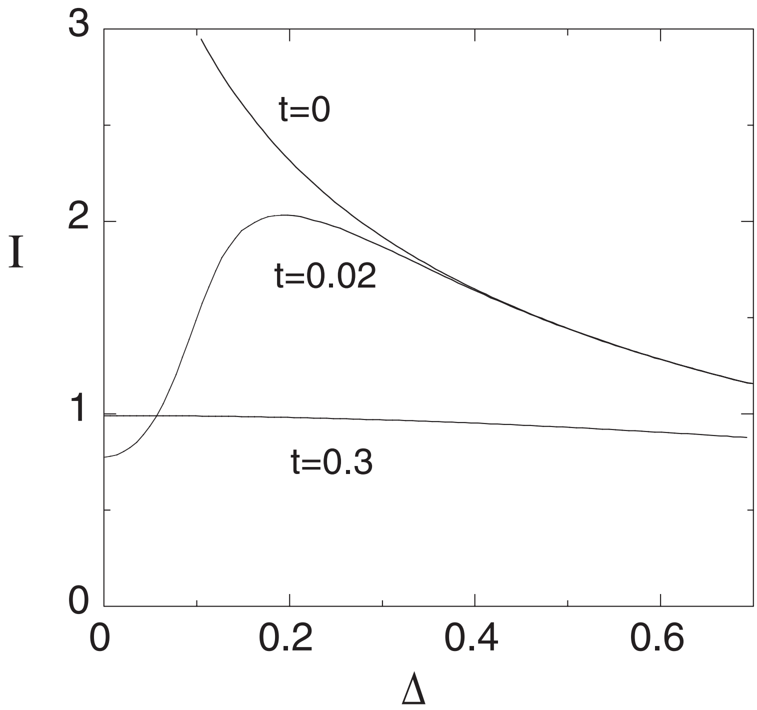

Fig. 7. The integral on the right-hand side of Eq. (5·16) as a function of Δ/ω_0 for $v/\omega_0 = 0.1$. Here we set $t/\omega_0 = 0$, 0.02 and 0.3 (from the top) and $\omega_0 = 1$.

and thus they are degenerate. In the original BCS model, the degeneracy is lifted. In this sense, supersymmetry is broken in the BCS Hamiltonian.

The BCS Hamiltonian possesses particle-hole symmetry. Let us examine this symmetry for the supersymmetric model. According to the particle-hole transformation, $\psi_{\rm BCS}$ and ψ_g are transformed into $\psi_{\rm BCS}^{\perp}$ and ψ_e, respectively, and vice versa. Then $\psi_{\rm BCS}^{\perp}$ and ψ_e become the ground states. Because $\psi_{\rm BCS}^{\perp}$ and ψ_e are not supersymmetric invariant, i.e. $Q^\dagger\psi_{\rm BCS}^{\perp} \neq 0$ and $Q\psi_e \neq 0$, the supersymmetry is broken in this case. Thus, we obtain a model for superconductivity with spontaneously broken supersymmetry after an electron-hole transformation.

The supersymmetric superconductivity displayed in this model is characterized by a peak in the density of states within the superconducting gap. We have presented a two-band model with supersymmetry. This system exhibits a transition from a superconducting state to an insulator, and vice versa, as the hybridization parameter v is varied. In the low temperature region, a first-order transition occurs, and in the ground state, this transition is at the supersymmetric point, $v = \Delta_0$. In the high temperature region, the transition becomes second order. It may be possible to adjust the parameter with some external forces, such as the pressure, in a two-band system in such a manner that a transition occurs across the supersymmetric point.

The two-band model possesses the dispersion relation of heavy-fermion systems described by the periodic Anderson model.[23),24)] In applying the present theory to real systems, the important problem is to determine the origin of the attractive

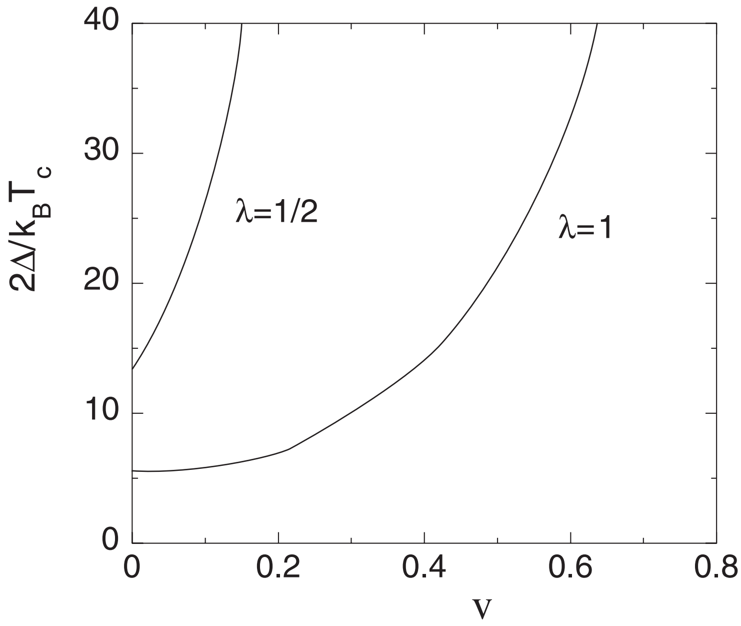

Fig. 8. The ratio $2\Delta(T=0)/k_BT_c$ as a function of v/ω_0 for $\lambda = 1$ and $1/2$.

interaction between the two bands. One type of phenomenon that could create such an attractive interaction is charge fluctuations, such as excitons or spin fluctuations, due to the interband interaction. If the origin of the attractive interaction is electronic, we could have a finite strength attractive interaction that is strong enough for the relation $v < \Delta_0$ to hold.

In summary, we have proposed a new supersymmetric model which exhibits an unusual superconductor-insulator first-order phase transition. The universal relations of the BCS theory, such as $\Delta(T)/\Delta(0)$ and $\Delta(0)/T_c$, do not hold in the present model.

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