Renormalization of the quantum antiferromagnet in two dimensions

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We discuss the renormalization of the quantum antiferromagnet in two dimensions. With an analysis of the spin-wave Hamiltonian of the quantum Heisenberg model or the quantum nonlinear σ model (NLσM), we consider $O(n)$ invariant observable $\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle$ and evaluate β functions in the dimensional regularization method up to two-loop order. We show that quantum and classical NL σ M are connected by a simple crossover function. It follows from this crossover phenomenon that the critical temperature T_c for $d>2$ is reduced due to the quantum effects and that the thermal fluctuations reduce the critical coupling constant g_c. The disappearance of g_c indicates thermal restoration of the asymptotic freedom. Finally we present an exact expression of the correlation length.

I. INTRODUCTION

The two-dimensional quantum antiferromagnet has received much attention since the discovery of the high-temperature superconductivity. The oxide compound shows a rich magnetic structure and it appears that quantum fluctuations play an essential role.[1-5] The undoped compound can be modeled rather well by a nearest-neighbor $S=\frac{1}{2}$ antiferromagnetic Heisenberg model

$$H=J\sum_{\langle ij\rangle} S_i S_j \ . \tag{1.1}$$

In the low-dimensional antiferromagnet, the difficulty in the straightforward application of the conventional spin wave theory lies in the divergencies encountered in the perturbation theory. These divergencies can be properly handled by the renormalization group method, which well succeeded in describing the properties of the classical spin systems near two dimensions.

The spin-wave theory of the antiferromagnetic Heisenberg model is closely related to the quantum nonlinear σ model (NLσM). To see this, we perform the standard two boson Dyson-Maleev transformation to convert the spin operators to the boson operators:[6,7]

$$S_i^- = a_i^\dagger, S_i^\dagger = (2S - a_i^\dagger a_i) a_i, S_i^z = S - a_i^\dagger a_i \quad \text{for the sublattice } A \ , \tag{1.2a}$$

$$S_m^- = -b_m, S_m^\dagger = -b_m^\dagger (2S - b_m^\dagger b_m)\ , S_m^z = -S + b_m^\dagger b_m \quad \text{for the sublattice } B \ . \tag{1.2b}$$

The Hamiltonian is written as follows:

$$H=-\tfrac{1}{2}JzS^2N + J\sum_{\langle im\rangle}[\,S(a_i^\dagger a_i + b_m^\dagger b_m - a_i^\dagger b_m^\dagger - a_i b_m) + \tfrac{1}{2}a_i^\dagger (b_m^\dagger - a_i)^2 b_m\,] \ . \tag{1.3}$$

Then let us expand the expectation value of $O(n)$ invariant spin operators in terms of $g_0 \equiv 2/S$,

$$\frac{1}{S^2}\langle \mathbf{S}_i\cdot\mathbf{S}_j\rangle = 1 + g_0(\langle a_i^\dagger a_j\rangle - \langle a_i^\dagger a_i\rangle) + \tfrac{1}{4}g_0^2(\langle a_i^\dagger a_j\rangle - \langle a_i^\dagger a_i\rangle)^2$$
$$= 1 + g_0\frac{2}{N}\sum_k \frac{1}{k}(e^{-ikR}-1)(n_k+\tfrac{1}{2}) + g_0^2\frac{1}{4}\left[\frac{2}{N}\sum_k\frac{1}{k}(e^{-ikR}-1)(n_k+\tfrac{1}{2})\right]^2 + O(g_0^3)\ , \tag{1.4}$$

for $i,j\in A$ where $\mathbf{R}=\mathbf{R}_i-\mathbf{R}_j$. The spin-wave excitation energy has been approximated by $\omega_k=2JSk$ and n_k is the Bose distribution function $n_k=1/(e^{\beta\omega_k}-1)$. The integral is performed over half of the first Brillouin zone. This expansion is identical with that obtained for NLσM. NLσM is the effective action defined by[8-10]

$$S=\frac{1}{2g_0}\int_0^\beta dx_0 \int d^dx (\partial_\mu \varphi)^2 \ , \tag{1.5}$$

where φ is an n-component field with the constraint $\varphi^2=1$. n is the multiplicity of spin components and $n=3$ for the spin-$\frac{1}{2}$ antiferromagnet. The inverse temperature β is defined by $\beta=g_0/t_0$ where t_0 is the scaled temperature $t_0=k_BT/(JS^2)$. The two-point function of φ field $\langle\varphi(0)\varphi(R)\rangle$ is given by

$$G(R)=1+(n-1)g_0[G^0(R)-G^0(0)] + \tfrac{1}{2}(n-1)g_0^2[G^0(R)-G^0(0)]^2 + O(g_0^3)\ , \tag{1.6}$$

where

$$G^0(R)=\frac{1}{\beta}\sum_{\omega}\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2+\omega^2}e^{-i\mathbf{p}\cdot\mathbf{R}}$$
$$=\int\frac{d^dp}{(2\pi)^d}\frac{1}{p}(n_p+\tfrac{1}{2})e^{-i\mathbf{p}\cdot\mathbf{R}} . \qquad (1.7)$$

Thus $G(R)$ in Eq. (1.6) clearly coincides with $\langle \mathbf{S}_i\cdot\mathbf{S}_j\rangle/S^2$ in Eq. (1.4), at least up to the order of g_0^2.

On the other hand, in the limit $t_0 \gg g_0$, which we call the classical region, the action (1.5) is reduced to the classical one, suggesting a crossover between classical and quantum systems:

$$S_{\rm cl}=\frac{1}{2t_0}\int d^dx(\partial_\mu\varphi)^2 . \qquad (1.8)$$

The coupling constant g_0 is replaced by the temperature t_0 in this region and thus we should expand $\langle\varphi(0)\varphi(R)\rangle$ in terms of t_0. This crossover is one of the subjects in this paper.

The renormalization-group approach to the spin-wave theory has been presented by several authors. Ridgway[11] first considered the one-dimensional quantum antiferromagnet and obtained the one-loop renormalization-group equations for $S=\frac{1}{2}$ at $T=0$ by the momentum-shell method following Wilson and Kogut.[12] Although the classical NLσM has been investigated considerably, the quantum NLσM has not yet examined so much. Chakravarty, Halperin, and Nelson[10] considered the NLσM also employing the momentum-shell method. They discussed the low-temperature behaviors of the correlation length based on the one-loop renormalization-group equations. Unfortunately, one fails to obtain two-loop contributions to the differential recursion relation in the momentum-shell method.

The purpose of this paper is to investigate a systematic expansion of the two-dimensional (2D) quantum antiferromagnet or $O(n)$ NLσM within the dimensional regularization scheme. We investigate β functions up to two-loop order in g and obtain the exact expression of the correlation length. We subsequently discuss a crossover between classical ($g_0 \ll t_0$) and quantum ($g_0 \gg t_0$) regions. Since the short-distance behaviors of the correlation functions are closely related to the g^2 term of the β functions, one can determine the crossover function. At one-loop order, the result coincides with that of Chakravarty, Halperin, and Nelson.[10]

The dimensional regularization method has been already applied to the one-dimensional antiferromagnet in Ref. 13, which predicted that the exponent η is different from that of the two-dimensional classical nonlinear σ model on the ground that the wave-function renormalization constant Z has no term of the order of g: $Z=1-(1/4\pi^2)g^2/\epsilon+O(g^3)$. However, this result seems strange to us, and really is not correct as shown later. The reason lies in the fact that in the renormalization of the low-dimensional quantum antiferromagnet we should consider $O(n)$ invariant quantities such as $\langle \mathbf{S}_i\cdot\mathbf{S}_j\rangle$, since the rotational symmetry is not broken. In Ref. 13 the author investigated N-point functions of α and β, which are not $O(n)$ invariant and not necessarily renormalizable. We show that at zero temperature the exponent η in the 1D antiferromagnet agrees with that of the 2D classical NLσM.

The format of the paper is as follows: In Sec. II we discuss the renormalization of the quantum antiferromagnet up to two-loop order. The crossover function is determined with a knowledge of the $1/n$ expansion. We discuss the crossover phenomenon by evaluating the critical temperature T_c. At the end of Sec. II we derive an exact expression of the correlation length ξ. In Sec. III we give a discussion of our results.

II. RENORMALIZATION OF THE QUANTUM ANTIFERROMAGNET

A. One-loop renormalization

We put $G_B(R_{ij})=\langle \mathbf{S}_i\cdot\mathbf{S}_j\rangle/S^2$ where $\mathbf{R}_{ij}=\mathbf{R}_i-\mathbf{R}_j$. At zero temperature $t=0$, $G_B(R_{ij})$ is given by

$$G_B(R_{ij})=1+(n-1)g_0[G^0(R_{ij})-G^0(0)]$$
$$+\tfrac{1}{2}(n-1)g_0^2[G^0(R_{ij})-G^0(0)]^2+O(g_0^3) , \qquad (2.1)$$

where n is the multiplicity of the spin components. In our formulation $G^0(R)$ is defined as the massless limit $\mu\to 0$ in the d-dimensional space

$$G^0(R)=\frac{1}{N}\sum_{\mathbf{k}}\frac{1}{\sqrt{k^2+\mu^2}}\frac{1}{2}e^{-i\mathbf{k}\cdot\mathbf{R}}$$
$$=\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{\sqrt{k^2+\mu^2}}e^{-i\mathbf{k}\cdot\mathbf{R}} . \qquad (2.2)$$

In the one-space dimension $G^0(R)$ has a divergence, which we regularize as a pole of $\epsilon=d-1$ as follows in the limit $\mu\to 0$:

$$G^0(R)=\frac{\Omega_d}{2(2\pi)^d}\frac{1}{\epsilon R^{\epsilon}} . \qquad (2.3)$$

It is understood that $G^0(R=0)\propto\mu^{\epsilon}\to 0$ as the standard dimensional regularization method requires.[14,15] Now it is easy to write the two-point function

$$G_B(R)=1+\tfrac{1}{2}(n-1)g_0\frac{1}{\epsilon R^{\epsilon}}$$
$$+\tfrac{1}{8}(n-1)g_0^2\frac{1}{\epsilon^2}\frac{1}{R^{2\epsilon}}+O(g_0^3) , \qquad (2.4)$$

where we have included a factor $\Omega_d/(2\pi)^d$ in our definition of g_0. We define the renormalized correlation function through the relation $G_R(R,g)=Z^{-1}G_B(R,g_0=Z_1g\mu^{1-d})$ where g is the renormalized coupling constant. The renormalization constants Z and Z_1 are determined as

$$Z^{-1}=1-\tfrac{1}{2}(n-1)\frac{1}{\epsilon}g+\tfrac{1}{8}(n-1)\frac{1}{\epsilon^2}g^2 , \qquad (2.5a)$$

$$Z_1=1+\tfrac{1}{2}(n-2)\frac{1}{\epsilon}g . \qquad (2.5b)$$

Then, recalling that β_g is related to Z_1 by the following relation:

$$\beta_g(\mu)=\mu\frac{dg}{d\mu}=\frac{\epsilon t}{1+g\frac{\partial}{\partial_g}\ln Z_1}\ , \tag{2.6}$$

we obtain the β function for the quantum region $g \gg t$,

$$\beta_g=\mu\frac{dg}{d\mu}=(d-1)g-\tfrac{1}{2}(n-2)g^2\ . \tag{2.7a}$$

At the critical point $g=g_c=2\epsilon/(n-2)$, the exponent η is given by

$$\eta=\beta_g\frac{\partial}{\partial g}\ln Z|_{g=g_c}-\epsilon=\frac{\epsilon}{n-2}+O(\epsilon^2)\ . \tag{2.8}$$

This value agrees with the exponent of the classical nonlinear σ model in two-space dimensions if we shift the dimension 1D to 2D. (Two-loop contributions will be shown later.) This is contrary to the claim given by Igarashi in Ref. 13. The difference between the two lies in the renormalization procedures; we have considered the $O(n)$ invariant quantity $\langle \mathbf{S}_i\cdot\mathbf{S}_j\rangle$, while in Ref. 13 the N-point functions $\Gamma^{(N)}$ are investigated. Since $\Gamma^{(N)}$ has no rotational invariance, its renormalizability is not clear. This point will be discussed in the last section.

Now we turn to the classical region $g \ll t$. In this limit the two-point function $G_B(R)$ is

$$G_B(R)=1+(n-1)t_0[G_c^0(R)-G_c^0(0)]+\tfrac{1}{2}(n-1)t_0^2[G_c^0(R)-G_c^0(0)]^2\ , \tag{2.9}$$

where the Green's function $G_c^0(R)$ is defined by

$$G_c^0(R)=\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}e^{-i\mathbf{k}\cdot\mathbf{R}}\ . \tag{2.10}$$

Thus it is apparent that we should regularize the divergence near two-space dimensions: $d=2+\epsilon$. Owing to the relation $G_R(R,t)=Z^{-1}G_B(R,t_0=Z_1t\mu^{2-d})$, renormalization constants are determined as follows:

$$Z^{-1}=1-(n-1)\frac{1}{\epsilon}t+\tfrac{1}{2}(n-1)\frac{1}{\epsilon^2}t^2\ , \tag{2.11a}$$

$$Z_1=1+(n-2)\frac{1}{\epsilon}t\ , \tag{2.11b}$$

where we have also included a factor $\Omega_d/(2\pi)^d$ in the definition of t. Then we obtain

$$\beta_t=\mu\frac{dt}{d\mu}=(d-2)t-(n-2)t^2\ , \tag{2.12a}$$

which is just the result of the classical nonlinear σ model. We have shown in the above that the renormalization of the correlation function (2.1) leads to classical and quantum results, respectively, for regions $t \gg g$ and $t \ll g$. In our opinion the two regions should be smoothly connected by a crossover function, which we will discuss in the following section. Before going to the next section we evaluate the β function of t for the quantum region and that of g for the classical region. Since the relation $g_0/t_0=(g/t)\mu^{-1}$ holds, we have $\beta_t=(t/g)\beta_g-t$. Hence in the quantum region we obtain

$$\beta_t=(d-2)t-\tfrac{1}{2}(n-2)gt\ , \tag{2.7b}$$

and next in the classical region

$$\beta_g=(d-1)g-(n-2)gt\ . \tag{2.12b}$$

B. Crossover function

In the following, we discuss the quantum to classical crossover. One may be able to pass from a quantum region to a classical one and back to a quantum one as g and t are varied. After the renormalization, the correlation function reads

$$G(R)=1-\tfrac{1}{2}(n-1)g\ln\mu R+\tfrac{1}{8}(n-1)(g^2+2\epsilon g)(\ln\mu R)^2 \quad \text{for } g \gg t \tag{2.13a}$$

$$=1-(n-1)t\ln\mu R+\tfrac{1}{2}(n-1)(t^2+\epsilon t)(\ln\mu R)^2 \quad \text{for } t \gg g\ . \tag{2.13b}$$

In order to obtain a unified expression for $G(R)$, we here consider it in a $1/n$ expansion. The leading order of $1/n$ theory leads us to an expression in two-space dimensions,

$$G(R)^{(1/n)}=ng\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{\sqrt{k^2+m^2}}\times\coth\left[\frac{g}{2t}\sqrt{k^2+m^2}\right]e^{-i\mathbf{k}\cdot\mathbf{R}}\ , \tag{2.14}$$

where $m=(2t/g)\sinh^{-1}\{\exp(-1/nt)\sinh(g/ntg_c)\}$ with the critical coupling constant $g_c=2/n$. [The volume element $\Omega_d/(2\pi)^d$ is included in g and t and a high-energy cutoff is set to be 1: $\Lambda=1$.] Then the short-distance correlation for $R \ll 1/m$ is given approximately by

$$G(R)^{(1/n)}\approx\frac{nt}{2\pi}\ln\left|\frac{\sinh(g/2tR)}{\sinh(gm/2t)}\right|\ . \tag{2.15}$$

This formula reproduces well Eqs. (2.13a) and (2.13b) for both limits up to the first order of $\ln R$. Expanding Eq. (2.15) in terms of $\ln R$, $G(R)^{(1/n)}\approx 1-\theta\ln R+\cdots$, one can easily derive the exponent θ,[16]

$$\theta=-\frac{(n-1)t}{2\pi}\frac{d\ln\sinh(g/2tR)}{d\ln R}\bigg|_{R=1}=\tfrac{1}{2}(n-1)g\coth\left[\frac{g}{2t}\right]\ . \tag{2.16}$$

(The multiplicity n has been replaced by an exact value $n-1$.) Evidently we can interpret $2\theta/(n-1)=g\coth(g/2t)$ as the effective coupling constant which depends on g and t. As a result we are allowed to write the correlation function as

$$G(R)=1-\tfrac{1}{2}(n-1)g\coth\left[\frac{g}{2t}\right]\ln\mu R+\tfrac{1}{8}(n-1)\left[g^2\coth\left[\frac{g}{2t}\right]^2+2\epsilon g\coth\left[\frac{g}{2t}\right]\right](\ln\mu R)^2\,. \quad (2.17)$$

One can note that this formula includes both limits of Eqs. (2.13a) and (2.13b).

We next turn to the β functions; they are written as

$$\beta_g=(d-1)g-\tfrac{1}{2}(n-2)g^2\coth\left[\frac{g}{2t}\right], \quad (2.18a)$$

$$\beta_t=(d-2)t-\tfrac{1}{2}(n-2)gt\coth\left[\frac{g}{2t}\right]. \quad (2.18b)$$

We simply discuss the crossover phenomenon by evaluating the critical temperature T_c. T_c is determined as the zero of β_t:

$$t_c\left[(d-2)-\tfrac{1}{2}(n-2)g\coth\left[\frac{g}{2t_c}\right]\right]=0\,. \quad (2.19)$$

In the limit $g\rightarrow 0$, t_c is given by $t_c=(d-2)/(n-2)\equiv t_c^0$ for $d>2$. In order to extract quantum corrections we expand Eq. (2.19) in powers of g. We will have

$$t_c=\frac{d-2}{n-2}-\frac{n-2}{12(d-2)}g^2+O(g^4)\,, \quad (2.20)$$

so that the critical temperature t_c is reduced due to the quantum effects. One may note that the quantum effects vanish in the limit of large d. In Fig. 1 we show t_c for $0\le g\le g_{\rm th}$. $g_{\rm th}$ means the threshold of t_c; at $g=g_{\rm th}=2t_c^0=2(d-2)/(n-2)$, t_c vanishes. This points out a breakdown of long-range order by the large quantum fluctuations. The similar story also holds for g_c. The equation of g_c is $\beta_g=0$:

$$g_c\left[d-1-\tfrac{1}{2}(n-1)g_c\coth\left[\frac{g_c}{2t}\right]\right]=0\,. \quad (2.21)$$

At low temperatures $t\ll g_c$, g_c is given by the well-known solution $g_c=2(d-1)/(n-2)\equiv g_c^0(d>1)$. In Fig. 2 we present g_c vs t for $0\le t\le t_{\rm th}\equiv(d-1)/(n-2)=g_c^0/2$. At the threshold $t=t_{\rm th}$, g_c disappears because of the thermal fluctuations. We may call this phenomenon the thermal reduction of g_c. In Figs. 3(a) and 3(b) we show the behaviors of β_g. For $1<d\le 2(t_c=0)$, we have two phases for $g<g_c$ and for $g>g_c$. For $g<g_c$, the infrared structure of the theory is determined by the origin; the mass gap vanishes as $t\rightarrow 0$. We call this phase the renormalized classical phase. The thermal reduction of g_c leads to the crossover around a critical value of t for which $g_c=g$. With further increase of t, g_c disappears leaving only an ultraviolet fixed point, and we have only the disordered phase. This observation shows up the restoration of asymptotic freedom. For $g>g_c$ we are in the disordered phase and have a finite gap. For $d>2$ there are three phases since we have finite t_c^0 and $g_{\rm th}$. For $g<g_{\rm th}(<g_c)$ and $t<t_c$, the theory exists in the Néel ordered phase. The renormalized classical phase corresponds to the case where $g<g_c$ and $t_c<t<t_{\rm th}$. Lastly, for $g>g_c$ and $t>t_c$, we have the disordered phase.

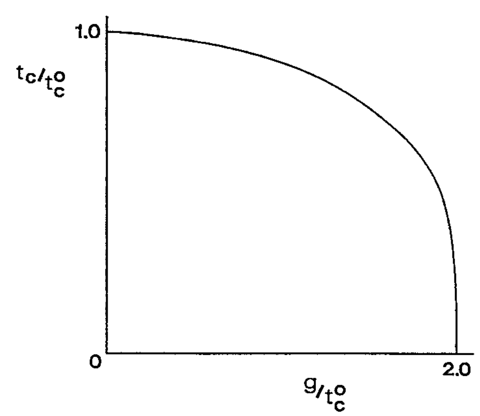

FIG. 1. The critical temperature t_c as a function of g.

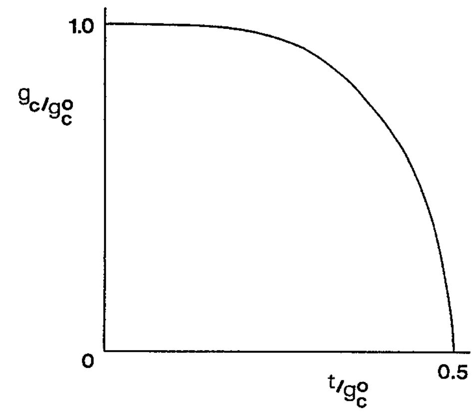

FIG. 2. The critical coupling constant g_c as a function of t.

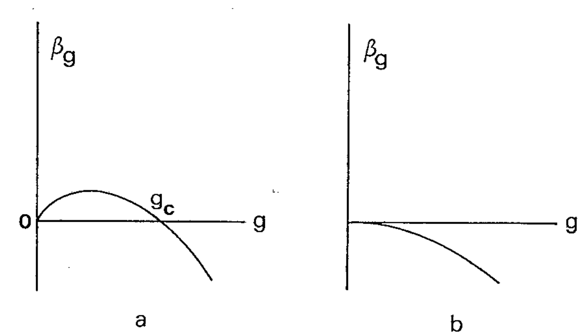

FIG. 3. The behaviors of β_g for (a) $t<t_{\rm th}$ and (b) $t\ge t_{\rm th}$ ($g_c=0$).

C. Two-loop renormalization

This section is devoted to evaluations of two-loop contributions. For this purpose it is convenient to work with the quantum nonlinear σ model defined as usual by the fields $\pi_i(i=1,\ldots,n-1)$ and $\sigma=(1-\pi^2)^{1/2}$:

$$S=\frac{1}{2g_0}\int_0^\beta dx_0\int d^2x\{(\partial_\mu\pi)^2+[\partial_\mu(1-\pi^2)^{1/2}]^2\}\ . \tag{2.22}$$

At the third order of g, the diagrams in Fig. 4 contribute to the correlation function, where the number of diagrams is greatly reduced due to the usual rule that $\int d^dk/k^\alpha=0$.[17,18] We present two-loop terms in Appendix A at $t=0$. The integrals are regularized in terms of $\epsilon=d-1$, so it is straightforward to write down the two-point function

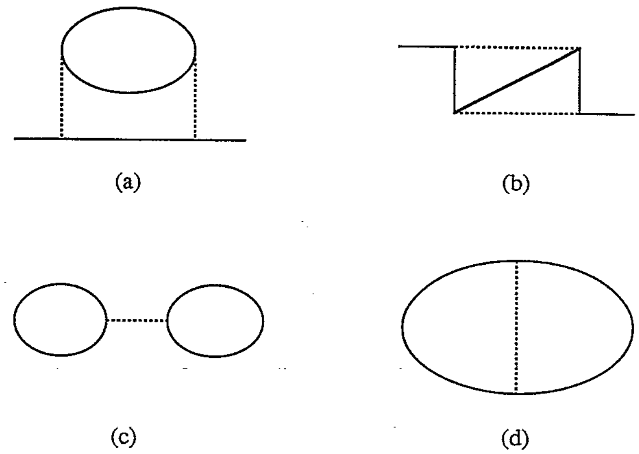

FIG. 4. Feynman diagrams for two-loop order perturbation. The dashed lines denote the interactions in the nonlinear σ model.

$$G_B(R)=1+\tfrac{1}{2}(n-1)g_0\frac{1}{\epsilon R^\epsilon}+\tfrac{1}{8}(n-1)g_0^2\frac{1}{\epsilon^2}\frac{1}{R^{2\epsilon}}$$
$$+\tfrac{1}{8}(n-1)g_0^3\left[-\tfrac{1}{6}(n-3)\frac{1}{\epsilon^3}-\tfrac{1}{6}(n-2)\frac{1}{\epsilon^2}+\tfrac{1}{4}(n-2)\frac{1}{\epsilon}+\tfrac{1}{6}(n-2)[\tfrac{3}{2}\zeta(3)-\tfrac{9}{4}]\right]\ . \tag{2.23}$$

Then the renormalization constants Z and Z_1 are determined easily and the β functions are obtained as

$$\beta_g=\epsilon g-\tfrac{1}{2}(n-2)g^2-\tfrac{1}{4}(n-2)g^3\ , \tag{2.24a}$$

$$\beta_t=(\epsilon-1)t-\tfrac{1}{2}(n-2)gt-\tfrac{1}{4}(n-2)g^2t\ . \tag{2.24b}$$

Following the discussions in the preceding section, the β functions for the whole range of g are given by

$$\beta_g=(d-1)g-\tfrac{1}{2}(n-2)g^2\coth\left[\frac{g}{2t}\right]-\tfrac{1}{4}(n-2)g^3\left[\coth\left[\frac{g}{2t}\right]\right]^2, \tag{2.25a}$$

$$\beta_t=(d-2)t-\tfrac{1}{2}(n-2)gt\coth\left[\frac{g}{2t}\right]-\tfrac{1}{4}(n-2)g^2t\left[\coth\left[\frac{g}{2t}\right]\right]^2. \tag{2.25b}$$

Up to $O(g^2,gt)$, these relations coincide with those of Chakravarty, Halperin, and Nelson and reproduce the well-known results in the classical region $g\ll t$.[19] It is also easily shown that the exponent η in the quantum region for $d=1+\epsilon$ agrees with that of the classical NLσM for $d=2+\epsilon$ up to two-loop order. The zero of β_t gives us the critical temperature t_c. For $g\to 0$, we have $t_c=t_c^0=[-1+\sqrt{1+4(d-2)/(n-2)}]/2$. One can easily observe that $t_c^0=0$ at $g=g_{\rm th}=2t_c^0$. This is the very same relation predicted by the one-loop theory in the preceding section. Hence certainly we can present the exact relation

$$g_{\rm th}=2t_c^0\quad\text{and}\quad t_{\rm th}=g_c^0/2\ . \tag{2.26}$$

Evidently the two-loop theory does not alter the story of the one-loop approximation.

D. Correlation length

Now let us turn to the correlation length ξ in this section on two-space dimensions $d=2(\epsilon=1)$ at low temperatures. The correlation length ξ is evaluated by integrating the β functions until the renormalized correlation length $(\mu/\Lambda)\xi$ is equal to the lattice constant. For $g<g_c$, $t(\mu)$ grows faster than $g(\mu)$ so that we choose μ^* such that $t(\mu^*)=1$. Then we obtain the correlation length by a formula $\xi=1/\mu^*$. It is well known that in the one-loop order ξ diverges exponentially by[10]

$$\xi=\frac{g}{2t}\exp\left[\frac{1}{n-2}\frac{1}{t}\left[1-\frac{g}{g_c^{(1)}}\right]\right], \tag{2.27}$$

as $T\to 0$ for $t\ll g<g_c^{(1)}$. Since in the classical region ξ is given by $\xi=\exp[1/(n-2)t]$, the factors $g/2t$ and $1-g/g_c^{(1)}$ in Eq. (2.27) are addressed to quantum corrections. The critical coupling constant $g_c^{(1)}$ is given by $2/(n-2)$ since we have put $\epsilon=1$. Up to two-loop order, Eqs. (2.25a) and (2.25b) can be integrated to obtain

$$\xi=\frac{1}{2}\left[\frac{g}{t}\right]^{1-1/(n-2)}\left[\frac{g}{f(g)}\right]^{1/(n-2)}\exp\left[\frac{1}{n-2}\frac{1}{t}f(g)\right]\exp\left[-\frac{1}{n-2}\ln\left[1+\frac{t}{f(g)}\right]\right],$$
$$=\frac{1}{2}\left[\frac{2\sqrt{2}JS}{k_BT}\right]^{1-1/(n-2)}\left[\frac{g}{f(g)}\right]^{1/(n-2)}\exp\left[\frac{1}{n-2}\frac{2\pi J}{k_BT}S^2f(g)\right]\left\{1-\frac{1}{n-2}\frac{k_BT}{2\pi J}\frac{1}{S^2}f(g)^{-1}\right\},\qquad(2.28)$$

as shown in Appendix B. The function $f(g)$ in (2.28) means $f(g)=(1-g/g_c^{(2)})^{\nu}(1-g\sqrt{\bar{g}_c}^{(2)})^{1-\nu}$, where $g_c^{(2)}$ (>0) and $\bar{g}_c^{(2)}$ (<0) are solutions of the quadratic equation $g_c^2+2g_c-4/(n-2)=0$. $g_c^{(2)}=-1+\sqrt{1+4/(n-2)}$ is the two-loop order critical coupling constant and ν is given by $\nu=(1-g_c^{(2)}\sqrt{\bar{g}_c}^{(2)})^{-1}$. A remarkable feature of the two-loop theory lies in the modifications of the coefficient of $1/t$ in the exponential function. In fact, formula (2.28) agrees well with the numerical integration of β_g as shown in Fig. 5 where $\ln\xi$ is plotted vs $x\equiv 1-g/g_c^{(2)}$ for $t=10^{-3}$. In Fig. 6 we present $\ln\xi$ as a function of temperature for $n=3$; clearly we can plot ξ in a form, $\ln\xi=c_1/t+c_2$. Equation (2.28) results in the expression of ξ for $S=\frac{1}{2}$ as

$$\xi=C_\xi\exp\left[\frac{2\pi J}{k_BT}F\right]\left[1-\frac{k_BT}{2\pi J}F^{-1}\right].\qquad(2.29)$$

C_ξ is a constant determined following the argument of Parisi:[20] $C_\xi=e^{c_2}\sqrt{32}e^{\pi/2}C_L$. Following the instanton calculus[21] $C_L=e^{1-\pi/2}/32\sqrt{2}$, we obtain an analytic expression $C_\xi=[g/2f(g)]e/8$. The constant F controls the temperature dependence of ξ at low temperatures. For the classical model constants are given by $F=S(S+1)$ and $C_\xi=e/8=0.3398$; on the other hand, for the quantum model the one-loop theory gives $F/S(S+1)=1-g/g_c=0.548$ and $C_\xi=0.279$, and next-order calculation predicts $F/S(S+1)=f(g)=0.44$ and $C_\xi=0.376$. In the estimation above we have used $g=\sqrt{d}/\pi S$. Thus we have determined C_ξ and F without any additional parameters and the values of F and C_ξ are consistent with the recent Monte Carlo calculations.[22–25]

With a large stock of the renormalization-group theory for the 2D classical nonlinear σ model,[26,27] we can easily expect the next-order term for β_g; it is given by $-(n-2)(n+2)/4g^4[\coth(g/2t)/2]^3$. This term produces only minor contributions to ξ:

$$\xi=\frac{1}{2}\left[\frac{g}{t}\right]^{1-1/(n-2)}\left[\frac{g}{f(g)}\right]^{1/(n-2)}\exp\left[\frac{1}{n-2}\frac{1}{t}f(g)\right]\times\left\{1-\frac{1}{n-2}\left[1-\frac{n+2}{4}\right]tf(g)^{-1}\right\}.\qquad(2.30)$$

The exponent $f(g)$ is controlled by the solutions of equation

$$g_c^3+[8/(n+2)]g_c^2+[16/(n+2)]g_c-32/(n-2)(n+2)=0.$$

Let three solutions be $g_c^{(3)}$, $z^{(3)}$, and $z^{(3)*}$ ($*$ refers to the complex conjugate). Then $f(g)$ is given by the simple expression

$$f(g)=(1-g/g_c^{(3)})^{\nu_1}|1-g/z^{(3)}|^{\nu_2}|1-g/z^{(3)*}|^{\nu_3},$$

where

$$\nu_1=1/(1-g_c^{(3)}/z^{(3)})(1-g_c^{(3)}/z^{(3)*}),$$
$$\nu_2=1/(1-z^{(3)}/g_c^{(3)})(1-z^{(3)}/z^{(3)*}),$$

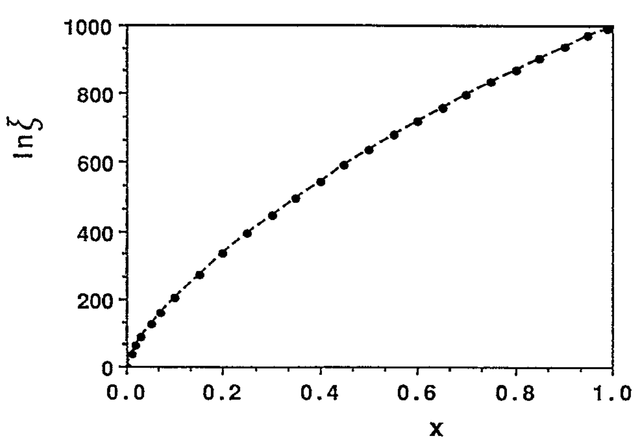

FIG. 5. $\ln\xi$ (closed circles) as a function of $x\equiv 1-g/g_c^{(2)}$ for $n=3$ and $t=10^{-3}$. The dashed line represents the two-loop function $t^{-1}f(g)\equiv t^{-1}(1-g/g_c^{(2)})^{\nu}(1-g\sqrt{\bar{g}_c}^{(2)})^{1-\nu}$.

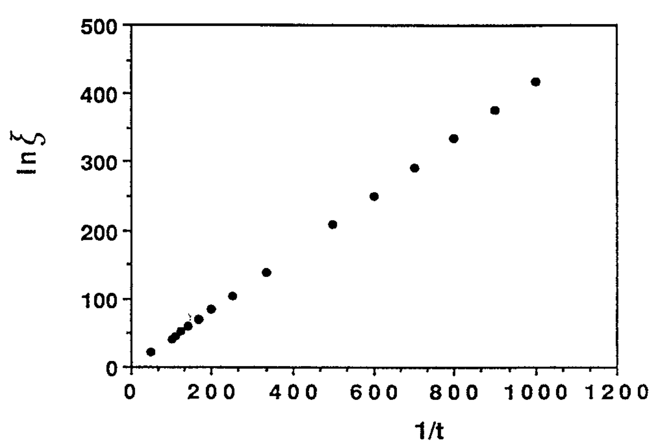

FIG. 6. $\ln\xi$ as a function of temperature t for $g=\sqrt{d}/\pi S=0.9003$.

TABLE I. Comparison of the values of F.

Method (Model)	F
Classical Heisenberg	$S(S+1)=0.75$
Quantum Heisenberg	$S(S+1)(1-g/gc)$
1 loop	$=0.41$
2 loop	0.33
3 loop	0.26
Monte Carlo (Ref. 22)	0.22($\pm$0.02)

and

$$\nu_3=\nu_2^*=1-\nu_1-\nu_2 .$$

For $n=3$, we obtain $F/S(S+1)=f(g)=0.35$, since $g_c^{(3)}=1.06193$ and $z^{(3)}=-1.33096+2.06284i$. We list the values of F in Table I for comparison. Our results well agree with the Monte Carlo calculations.[22]

III. DISCUSSIONS

In this paper we have discussed the renormalization of the quantum antiferromagnet in low-dimensional space. We have obtained the renormalization constants and the recursion relations up to two-loop order within the dimensional regularization method rather than using the momentum-shell one. In the momentum-shell method, one fails to derive the differential recursion relation to the two-loop order. We have two distinct regions called the quantum region for $t \ll g$ and the classical region for $t \gg g$. In the classical region the β functions agree with those of the classical nonlinear σ model at two-space dimensions. We have shown that the interplay between two regions is described by the simple crossover function $\coth(g/2t)$.

The fundamental problem in the low-dimensional antiferromagnets lies in the observation that the ordinary perturbation theory is a perturbation about a vacuum with a broken continuous symmetry. This difficulty emerges as the infrared divergencies in the perturbation expansions. In field theory the infrared divergence has some connection with color confinement or symmetry restoration. Recently the spin-wave theory with a slight modification has been applied successfully to antiferromagnets as well as ferromagnets in two dimensions at low temperatures.[28] This demonstrates the possibility that one can study the perturbation expansion around the wrong vacuum, with symmetry restored dynamically. Since the rotational symmetry is not broken, we are able to consider rotational invariant quantities. In this paper, regularizing the divergences of $\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle$ by the dimensional scheme, we have obtained the β functions up to two-loop order. A rich knowledge of the 2D classical NLσM teaches us the third-order term. We have derived the formula for the critical temperature t_c to show that t_c is reduced by quantum fluctuations. t_c has a solution for $0 \le g \le g_{\rm th}=2t_c^0$ and vanishes at $g=g_{\rm th}$. The relation $g_{\rm th}=2t_c^0$ holds in each order of perturbation theory and consequently can be an exact relation. We have also discussed the thermal reduction of g_c. The threshold value of t is given by $t_{\rm th}=g_c^0/2; g_c=0$ at $t=g_c^0/2$. What is remarkable is that our model shows the restoration of the asymptotic freedom at $t=t_{\rm th}$ for $d>1$. The quantum model includes a rather richer structure than the classical one.

We have derived the expression of correlation length at low temperatures up to two- and three-loop order without free parameters. ξ is written in a form $\xi=C_\xi \exp(2\pi JF/k_B T)$, where the constant F is factorized as a classical contribution F_c times quantum corrections: $F=F_c f(g)$. F and C_ξ are consistent with the results of Monte Carlo simulations.

Finally, let us discuss the renormalization of the two-point function $\Gamma^{(2)}$. In order to diagonalize the quadratic part of the Dyson-Maleev Hamiltonian, we introduce the Bogoliubov transformation

$$a_k=u_k\alpha_k-v_k\beta_{-k}^\dagger , \tag{3.1a}$$

$$b_{-k}^\dagger=u_k\beta_{-k}^\dagger-v_k\alpha_k . \tag{3.1b}$$

Multiplying the renormalization constant Z_φ,

$$\alpha_{k,R}^\dagger=Z_\varphi^{-1/2}\alpha_{k,B}^\dagger, \beta_{k,R}^\dagger=Z_\varphi^{-1/2}\beta_{k,B}^\dagger , \tag{3.2}$$

we define the renormalized N-point function,

$$\Gamma_R^{(N)}(\mathbf{k},\omega,g\mu)=Z_\varphi^{-N/2}\Gamma_B^{(N)}(\mathbf{k},\omega,g_0) . \tag{3.3}$$

Here we consider the two-loop function $\Gamma_\alpha^{(2)}(\mathbf{k},\omega)$ of α. It was shown in Ref. 11 that the self-energy corrections for $\Gamma_\alpha^{(2)}$ vanish to one-loop order. Hence a nontrivial contribution to $\Gamma_\alpha^{(2)}$ is of the order of g^2:

$$\Gamma_{\alpha,B}^{(2)}(\mathbf{k},\omega)=\omega\left[1+Ag^2\frac{1}{\epsilon}\right] \quad \text{for } \omega \gg |\mathbf{k}| , \tag{3.4}$$

where A is a constant. This result implies that Z_φ is given by $Z_\varphi=1-Ag^2/\epsilon$, which is different from our Z. This is clearly incorrect. Thus we should consider $\langle \alpha_k^\dagger \alpha_k+\beta_{-k}\beta_{-k}^\dagger \rangle$ to obtain correct results.

APPENDIX A

In this appendix, we show terms contributing to the two-loop renormalization. They are listed in Fig. 3 in the momentum space. If we use a notation $p=(p_0,\mathbf{p})$, where $p_0=2\pi n/\beta$ (n is an integer), the Feynman diagram in Fig. 1(a) at $t=0$ for $d=1+\epsilon$ is calculated as

$$\frac{1}{\beta}\sum_{q_0}\int\frac{d^dp}{(2\pi)^d}\frac{1}{\beta}\sum_{k_0}\int\frac{d^dp}{(2\pi)^d}\frac{(p+q)^4}{q^2k^2(p+q+k)^2} = \left[\frac{\Omega_d}{2(2\pi)^d}\right]^3\frac{1}{\epsilon^2}\left[1+\frac{\epsilon}{2}\right]\frac{1}{\pi}\Gamma\left[\frac{1+\epsilon}{2}\right]^2 \times\Gamma\left[1+\frac{\epsilon}{2}\right]^3\Gamma(1-\epsilon)\frac{1}{\Gamma(2+3\epsilon/2)}(p^2)^{1+\epsilon} . \tag{A1}$$

After a summation of p_0, we can easily transform this expression to the real-space representation. The results are listed below:

$$(a)\ \frac{1}{2}(n-1)^2g^3\left[\frac{\Omega_d}{2(2\pi)^d}\right]^3\frac{1}{\epsilon^3R^{3\epsilon}}\frac{1}{3}\left[1+\frac{\epsilon}{2}\right]\frac{1}{\pi}\Gamma\left[\frac{1+\epsilon}{2}\right]\Gamma\left[1+\frac{\epsilon}{2}\right]^3\Gamma(1-2\epsilon)\Gamma(1+3\epsilon)\frac{1}{\Gamma(1-\epsilon)}\frac{1}{\Gamma(2+3\epsilon/2)}, \tag{A2}$$

$$(b)\ (n-1)g^3\left[\frac{\Omega_d}{2(2\pi)^d}\right]^3\frac{1}{\epsilon^2R^{3\epsilon}}\frac{1}{6}\frac{1}{\pi}\Gamma\left[\frac{1+\epsilon}{2}\right]^2\Gamma\left[1+\frac{\epsilon}{2}\right]^3\Gamma(1-2\epsilon)\Gamma(1+3\epsilon)\frac{1}{\Gamma(1-\epsilon)}\frac{1}{\Gamma(2+3\epsilon/2)}, \tag{A3}$$

$$(c)\ -\tfrac{1}{4}(n-1)^2g^3\left[\frac{\Omega_d}{2(2\pi)^d}\right]^3\frac{1}{\epsilon^3R^{3\epsilon}}\frac{4}{3}\frac{1}{\pi}\Gamma\left[\frac{1+\epsilon}{2}\right]^2\Gamma\left[1+\frac{\epsilon}{2}\right]^4\Gamma\left[1-\frac{\epsilon}{2}\right]^2\Gamma(1-2\epsilon)\Gamma(1+3\epsilon)\frac{1}{\Gamma(1-\epsilon)^2}\frac{1}{\Gamma(1+\epsilon)^2}, \tag{A4}$$

$$(d)\ -\tfrac{1}{2}(n-1)g^3\left[\frac{\Omega_d}{2(2\pi)^d}\right]^3\frac{1}{\epsilon^3R^{3\epsilon}}\frac{2}{3}\frac{1}{\pi}\Gamma\left[\frac{1+\epsilon}{2}\right]^2\Gamma\left[1+\frac{\epsilon}{2}\right]^4\Gamma\left[1-\frac{\epsilon}{2}\right]^2\Gamma(1-2\epsilon)\Gamma(1+3\epsilon)\frac{1}{\Gamma(1-\epsilon)^2}\frac{1}{\Gamma(1+\epsilon)^2}. \tag{A5}$$

APPENDIX B

Introducing μ_1 such that $\mu_1\approx 2/\beta$, let us write the recursion equations for $d=2$ as follows:

$$\mu\frac{dt}{d\mu}=-\tfrac{1}{2}(n-2)\beta t^2\mu-\tfrac{1}{4}(n-2)\beta^2t^3\mu^2 \quad \text{for } \mu>\mu_1, \tag{B1a}$$

and

$$\mu\frac{dt}{d\mu}=-(n-2)t^2-(n-2)t^3 \quad \text{for } \mu<\mu_1, \tag{B1b}$$

where β is the initial value of $g(\mu)/t(\mu)$ at $\mu=\Lambda\equiv 1$: $\beta\equiv g(\mu=\Lambda)/t(\mu=\Lambda)\equiv g_0/t_0=2\sqrt{d}JS/k_BT$. Equation (B1a) easily results in a formula,

$$\ln(\tfrac{1}{2}\beta\mu)=-\frac{1}{\alpha_1-\alpha_2}\left[\alpha_1\ln\left[\frac{2}{t\beta\mu}-\alpha_1\right]-\alpha_2\ln\left[\frac{2}{t\beta\mu}-\alpha_2\right]\right]+C, \tag{B2}$$

with an integral constant C. α_1 and α_2 are solutions of the quadratic equation $x^2-(n-2)(x+1)=0$:

$$\alpha_1=[n-2+\sqrt{(n-2)^2+4(n-2)}]$$

and

$$\alpha_2=[n-2-\sqrt{(n-2)^2+4(n-2)}].$$

The inverse of α_1 gives the critical coupling constant $g_c^{(2)}=2/\alpha_1$ and we put $\bar{g}_c^{(2)}\equiv 2/\alpha_2$. Due to the initial condition at $\mu=1$, C is given by

$$C=\ln\left[\frac{\beta}{2}\right]+\frac{1}{\alpha_1-\alpha_2}\left[\alpha_1\ln\left[\frac{2}{g_0}-\alpha_1\right]-\alpha_2\ln\left[\frac{2}{g_0}-\alpha_2\right]\right]. \tag{B3}$$

If we set $t=t_1\ll 1$ at $\mu=\mu_1$, we obtain

$$\frac{1}{t_1}=e^C=\beta\left[\frac{1}{g_0}-\frac{1}{g_c^{(2)}}\right]^\nu\left[\frac{1}{g_0}-\frac{1}{\bar{g}_c^{(2)}}\right]^{1-\nu}, \tag{B4}$$

where $\nu=\alpha_1/(\alpha_1-\alpha_2)$. Next let us turn to a solution of Eq. (B1b). In order to obtain the correlation length ξ, we choose μ^* such that $t(\mu^*)=1$. We demand that the solutions of Eqs. (B1a) and (B1b) coincide at $\mu=\mu_1$; we have

$$\begin{aligned}\xi=\frac{1}{\mu^*}&=\frac{\beta}{2}\exp\left[\frac{1}{n-2}\left[\frac{1}{t_1}+\ln t_1-\ln(1+t_1)\right]\right]\\&=\frac{1}{2}\left[\frac{g_0}{t_0}\right]^{1-1/(n-2)}\left[\frac{g_0}{f(g_0)}\right]^{1/(n-2)}\\&\quad\times\exp\left[\frac{1}{n-2}\frac{1}{t_0}f(g_0)\right]\\&\quad\times\exp\left[-\frac{1}{n-2}\ln\left[1+\frac{t_0}{f(g_0)}\right]\right].\end{aligned} \tag{B5}$$

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