Theory of Multi-Band Superconductivity

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Multi-band superconductors exhibit many interesting and novel properties. Most of superconductors with high critical temperatures are multi-band superconductors; for instance, MgB_2, iron-based superconductors and layered cuprates. The study of multi-band superconductors stemmed from the works by J. Kondo and Suhl et al., as the generalization of the Bardeen-Cooper-Schrieffer (BCS) theory to the case with two conduction bands.

Multi-gap systems have additional phase invariance, compared to the single-gap superconductors, which will bring about novel characteristics. One of most important subjects is dynamics of the phase difference mode, that is, fluctuation of the phase difference between two gaps. The phase difference mode is called the Leggett mode or Leggett-Kondo mode. This mode yields half-quantum flux vortices in two-gap superconductors. Half-quantized flux vortices exist in two-gap superconductors under a magnetic field. Two half-quantum flux vortices form a bound state connected by a domain wall. A generalization to three-gap superconductors is very attractive; the superconducting state is a chiral state with time-reversal symmetry breaking and fractionally quantized flux vortices exist. We can formulate vortices with 1/3-quantum flux in three-component superconductors with equivalent three bands. A three-vortex bound state is formulated and this bound state resembles a baryon in QCD. Two or three fractionally quantized vortices are connected by domain walls and form a bound state.

Lastly we discuss that the Leggett mode can be represented as a gauge field. The gauge invariance of a multi-band model should be certainly discussed with the Leggett mode. This is closely related with $SU(N)$ gauge theory, especially its abelian projection.

I Introduction

Multi-band superconductors are very attractive because of many interesting and novel phenomena that are not found in single-band supercondutors. The study of multi-band superconductors started, as a generalization of the BCS theory[1] to multi-gap models, from works by Suhl et al.[2] and Kondo[3]. Kondo pointed out that the sign of gap function depends on the sign of the pair-transfer interaction in two-band superconductors, and the signs of two gaps are opposite to each other when the pair-transfer interaction is repulsive. Main characteristics of multi-band superconductors are the following:

(1) In general, the critical temperature T_c is high in multi-band superconductors. MgB_2[4] and iron-based superconductors[5] are well known, and layered cuprates[6] are also regarded as multi-gap superconductors.

(2) There are new interesting properties in multi-band superconductors. For example, the isotope exponent α takes values even in the range of $\alpha < 0$ and $\alpha > 0.5$, depending on the strength and the range of attractive interactions[7, 8].

(3) Multi-phase physics is a new physics of multi-band superconductors. An additional phase invariance will bring about novel phenomena. The phase difference mode between two gaps is called the Leggett mode[9] or Leggett-Kondo mode. The Leggett mode will yield new excitation modes in multi-band superconductors.

(4) The existence of fractionally quantized flux vortices is very significant and attractive. A kink solution leads to a half-quantum flux vortex in two-gap superconductors[10, 11]. A generalization to a three-gap superconductor results in very attractive features, that is, chiral states with time-reversal symmetry breaking and the existence of fractionally quantized vortices[12–15]. Further, in the case with more than four gaps, a new state is predicted with a gapless excitation mode[16].

(5) A new type of superconductors, called the 1.5 type as an intermediate of types I and II, has been proposed for two-gap superconductors[17]. This state may be realized as a result of a multi-band effect, and does not occur in a single-band superconductor.

(6) There is an interesting and profound analogy between particles physics and superconductivity. The mass of the Higgs particle corresponds to the inverse of the coherence length, and the masses of gauge bosons W and Z correspond to the inverse of the penetration depth. If we use $m_W \sim 80.41\text{GeV}/c^2$, $m_Z \sim 91.19\text{GeV}/c^2$ and $m_H \sim 126\text{GeV}/c^2$, the Ginzburg-Landau parameter κ is roughly

$$\kappa = \frac{\lambda}{\xi} \sim \frac{m_{W,Z}}{m_H} \sim 1.5. \tag{1}$$

This indicates that the universe corresponds to a (multi-component) type-II superconductor.

The global $U(1)$ phase invariance is spontaneously broken in superconductors. We say that spontaneous symmetry breaking occurred when the ground state of the system lost the invariance of the Hamiltonian of the system. There is no reason why an invariance of the Hamiltonian should be an invariance of the ground state. A good basic example is the Heisenberg model with spin-spin exchange interactions between nearest neighbors. The ground state is not rotationally invariant, with spins aligned in one direction, although the Hamiltonian is rotationally invariant. The ground state has a long-range order by breaking the rotational invariance. It is well known that the gapless Goldstone mode exists when the continuous symmetry is spontaneously broken. In the

Heisenberg ferromagnet, this gapless mode is the spin wave excitation. There are many models that exhibit spontaneous symmetry breaking in the condensed-matter physics.

Superconductivity is most familiar phenomenon that occurs as a result of spontaneous symmetry breaking. This is described by the Ginzburg-Landau functional[18]:

$$F = \int d^3x \Big[\alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{\hbar^2}{2m}\Big|\left(\nabla + i\frac{2\pi}{\phi_0}\mathbf{A}\right)\psi\Big|^2 + \frac{1}{8\pi}H^2\Big]. \tag{2}$$

ψ is the order parameter, $\beta > 0$ and α is written as $\alpha = \alpha_0(T - T_c)$. This free energy describes a spontaneous breaking of $U(1)$ symmetry. When $\alpha < 0$, that is $T < T_c$, the minimization of F yields a solution $\psi \neq 0$. The potential

$$V(\psi) = \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4, \tag{3}$$

has an infinite number of possible minima for $\alpha < 0$ given by

$$\psi = \sqrt{-\frac{\alpha}{\beta}}e^{i\theta} \tag{4}$$

for any real angle θ in the range $0 \leq \theta \leq 2\pi$. Any choice of θ would have exactly the same energy that implies the existence of a massless Nambu-Goldstone boson. This changes qualitatively when the Coulomb interaction between the electrons is included. The Coulomb repulsive interaction turns the massless mode into a gapped plasma mode[19]. Therefore the mode that originates from the phase variable θ does not play an important role. This would change qualitatively in multi-band superconductors because the multi-phase variables, namely, the Leggett mode variables will produce new excited states.

It is well known that there is an interesting analogy between superconductivity and particle physics. We show a list of correspondences between superconductivity and particle physics in Table I. The gapless mode emerging from the spontaneously broken global $U(1)$ symmetry in superconductors is called the Anderson-Bogoliubov mode. TRS breaking denotes time-reversal symmetry breaking which will occur in a triplet-pairing superconductor or in a three-band superconductor. Here we have a question: what is the counterpart of the Leggett-Kondo mode in particle physics? Our answer is that: the Leggett-Kondo mode can be represented as a gauge field with diagonal elements, and therefore would correspond to an abelian projection of the $SU(N)$ gauge theory[20]. This will be discussed in detail below.

TABLE I:

Superconductors	Particles
Broken symmetry	Broken symmetry
Ground state: BCS	Vacuum
Energy gap	Mass
Quasi particles	Particles
Gapless mode (Anderson-Bogoliubov)	Nambu-Goldstone mode
Plasma mode	Higgs mechanism
Meissner effect	Higgs mechanism
TRS breaking	CP violation
Half-quantum vortex	Monopole
Fractional vortices	Quarks
Leggett mode	Abelian projection

II The BCS theory

Let us consider the BCS Hamiltonian:

$$H = \int d\mathbf{r}\sum_\sigma \psi_\sigma^\dagger(\mathbf{r})\left(\frac{\mathbf{p}^2}{2m} - \mu\right)\psi_\sigma(\mathbf{r}) - g\int d\mathbf{r}\psi_\uparrow^\dagger(\mathbf{r})\psi_\downarrow^\dagger(\mathbf{r})\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r}), \tag{5}$$

where σ is the spin index $\uparrow$ and $\downarrow$, μ is the chemical potential and $g > 0$ is the coupling constant of the attractive interaction. In the momentum space, this is written as

$$H = \sum_{k\sigma}\xi_k c_{k\sigma}^\dagger c_{k\sigma} - g\frac{1}{V}\sum_{kk'q} c_{k'\uparrow}^\dagger c_{-k'+q\downarrow}^\dagger c_{-k+q\downarrow}c_{k\uparrow}, \tag{6}$$

where $\xi_k = \epsilon_k - \mu$ for the electron dispersion ϵ_k. The corresponding Lagrangian density is

$$\mathcal{L} = \sum_\sigma \psi_\sigma^\dagger(x)\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m} + \mu\nabla^2\right)\psi_\sigma(x) + g\psi_\uparrow^\dagger(x)\psi_\downarrow^\dagger(x)\psi_\downarrow(x)\psi_\uparrow(x). \tag{7}$$

Using the Nambu notation[21],

$$\psi = \begin{pmatrix}\psi_\uparrow \\ \psi_\downarrow^\dagger\end{pmatrix}, \tag{8}$$

the Lagrangian density becomes

$$\mathcal{L} = \psi^\dagger\left(\sigma_0 i\hbar\frac{\partial}{\partial t} - \sigma_3\xi(\nabla)\right)\psi - \frac{g}{4}\left[(\psi^\dagger\psi)^2 - (\psi^\dagger\sigma_3\psi)^2\right], \tag{9}$$

where σ_0 is the unit matrix and $\xi(\nabla) = -\hbar^2\nabla^2/(2m) - \mu = \mathbf{p}^2/(2m) - \mu$. The vacuum partition function is represented by a functional integral,

$$Z = \int d\psi^\dagger d\psi \exp\left(\frac{i}{\hbar}\int d^dx\mathcal{L}\right). \tag{10}$$

d is the space-time dimension. This can be written in a bilinear form by applying a Hubbard-Stratonovich transformation,

$$\exp\left(\frac{i}{\hbar}g\int d^dx\psi_\uparrow^\dagger\psi_\downarrow^\dagger\psi_\downarrow\psi_\uparrow\right)$$
$$=\int d\Delta^* d\Delta\exp\Bigg[-\frac{i}{\hbar}\int d^dx\Big[\Delta^*\psi_\downarrow\psi_\uparrow+\Delta\psi_\uparrow^\dagger\psi_\downarrow^\dagger$$
$$+\frac{1}{g}|\Delta|^2\Big]\Bigg], \qquad (11)$$

where Δ^* and Δ are auxiliary fields and an overall normalization factor is excluded. The partition function has the form

$$Z=\int d\psi^\dagger d\psi\int d\Delta^* d\Delta\exp\left(\frac{i}{\hbar}\int d^dx\mathcal{L}_{eff}\right), \qquad (12)$$

where

$$\mathcal{L}_{eff}=\psi^\dagger\Big[\sigma_0 i\hbar\frac{\partial}{\partial t}-\sigma_3\xi(\nabla)-\begin{pmatrix}0&\Delta\\ \Delta^*&0\end{pmatrix}\Big]\psi-\frac{1}{g}|\Delta|^2. \qquad (13)$$

The field equations obtained by variation of the Lagrangian are

$$\left[i\hbar\frac{\partial}{\partial t}-\sigma_3\xi(\nabla)-\begin{pmatrix}0&\Delta\\ \Delta^*&0\end{pmatrix}\right]\psi=0, \qquad (14)$$

$$\Delta=g\psi_\uparrow\psi_\downarrow. \qquad (15)$$

The equation for Δ shows that Δ describes a pair of electrons that forms a spin-singlet. If we approximate Δ by its average $\bar{\Delta}=g\langle\psi_\uparrow\psi_\downarrow\rangle$, we obtain a self-consistency equation for $\bar{\Delta}$. By performing the Grassman integration over the fields $\psi^\dagger$ and ψ, we obtain the effective action

$$S(\Delta^*,\Delta) = -\frac{1}{g}\int d^dx|\Delta(x)|^2$$
$$- i\hbar\mathrm{Tr}\ln\begin{pmatrix}p_0-\xi(\mathbf{p})&-\Delta(x)\\ -\Delta^*(x)&p_0+\xi(\mathbf{p})\end{pmatrix}, \qquad (16)$$

for which the partition function is

$$Z=\int d\Delta^* d\Delta\exp\left(\frac{i}{\hbar}S(\Delta^*,\Delta)\right). \qquad (17)$$

Now the averaged value $\bar{\Delta}$ of the gap function Δ is determined by adopting the saddle point approximation. The field equation reads

$$\frac{\delta S(\bar{\Delta}^*,\bar{\Delta})}{\delta\bar{\Delta}^*}=0. \qquad (18)$$

We obtain a solution assuming that $\bar{\Delta}>0$ is a constant. This yields

$$\frac{1}{g}\bar{\Delta}=i\hbar\mathrm{Tr}G_0(p)\begin{pmatrix}0&0\\1&0\end{pmatrix}, \qquad (19)$$

where $G_0(p)$ is the Green function including $\bar{\Delta}$,

$$G_0(p) = \begin{pmatrix}p_0-\xi(\mathbf{p})&-\bar{\Delta}\\ -\bar{\Delta}^*&p_0+\xi(\mathbf{p})\end{pmatrix}^{-1}$$
$$= \frac{1}{p_0^2-E(\mathbf{p})^2+i\delta}\begin{pmatrix}p_0+\xi(\mathbf{p})&\bar{\Delta}\\ \bar{\Delta}^*&p_0-\xi(\mathbf{p})\end{pmatrix}. \qquad (20)$$

Here,

$$E(\mathbf{p})=\sqrt{\xi(\mathbf{p})^2+\bar{\Delta}^2} \qquad (21)$$

is the single-particle excitation energy. Then we obtain the gap equation

$$\frac{1}{g}=\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{E(\mathbf{k})}. \qquad (22)$$

The superconducting gap is

$$\bar{\Delta}=2\hbar\omega_D\exp\left(-\frac{1}{\rho g}\right), \qquad (23)$$

with the energy cutoff $\hbar\omega_D$ and the density of states ρ at the Fermi energy.

III Multi-Band Superconductors

When we consider a multi-band superconductor, the pair-transfer interactions between different bands (Josephson couplings) are introduced to the Hamiltonian. This type of interaction was examined Suhl et al.[2] and Kondo[3] for the two-band model. This interaction was called the exchange-like integral by Kondo, and it is expected that this interaction will enhance the critical temperature T_c[22, 23].

The multi-band BCS model with the attractive interactions is given as

$$H=\sum_{i\sigma}\int d\mathbf{r}\psi_{i\sigma}^\dagger(\mathbf{r})K_i(\mathbf{r})\psi_{i\sigma}(\mathbf{r})$$
$$-\sum_{ij}g_{ij}\int d\mathbf{r}\psi_{i\uparrow}^\dagger(\mathbf{r})\psi_{i\downarrow}^\dagger(\mathbf{r})\psi_{j\downarrow}(\mathbf{r})\psi_{j\uparrow}(\mathbf{r}), \qquad (24)$$

where i and j (=1,2,$\cdots$) are band indices. $K_i(\mathbf{r})$ stands for the kinetic operator. We assume that $g_{ij}=g_{ji}^*$. The second term is the pairing interaction and g_{ij} are coupling constants. The mean-field Hamiltonian is

$$H_{MF}=\sum_i\int d\mathbf{r}\Big[\sum_\sigma\psi_{i\sigma}^\dagger(\mathbf{r})K_i(\mathbf{r})\psi_{i\sigma}(\mathbf{r})$$
$$+\Delta_i(\mathbf{r})\psi_{i\uparrow}^\dagger(\mathbf{r})\psi_{i\downarrow}^\dagger(\mathbf{r})+\Delta_i^*(\mathbf{r})\psi_{i\downarrow}(\mathbf{r})\psi_{i\uparrow}(\mathbf{r})\Big], \qquad (25)$$

where the gap function in each band is defined by

$$\Delta_i(\mathbf{r})=-\sum_j g_{ij}\langle\psi_{j\downarrow}(\mathbf{r})\psi_{j\uparrow}(\mathbf{r})\rangle, \qquad (26)$$

and its complex conjugate is

$$\Delta_i^*(\mathbf{r}) = -\sum_j g_{ji}\langle\psi_{j\uparrow}^\dagger(\mathbf{r})\psi_{j\downarrow}^\dagger(\mathbf{r})\rangle. \tag{27}$$

We define Green's functions as follows[24],

$$G_{j\sigma\sigma'}(x-x') = -\langle T_\tau\psi_{j\sigma}(x)\psi_{j\sigma'}^\dagger(x')\rangle, \tag{28}$$

$$F_{j\sigma\sigma'}^+(x-x') = \langle T_\tau\psi_{j\sigma}^\dagger(x)\psi_{j\sigma'}^\dagger(x')\rangle, \tag{29}$$

where T_τ is the time-ordering operator and we use the notation $x = (\tau, \mathbf{r})$. In terms of the Green's functions, the gap functions satisfy the system of equations

$$\begin{aligned}\Delta_i^*(\mathbf{r}) &= \sum_j g_{ij}^* F_{j\downarrow\uparrow}^+(\tau' = \tau+0; \mathbf{r}, \mathbf{r}) \\ &= \sum_j g_{ij}^* \frac{1}{\beta}\sum_n F_{j\downarrow\uparrow}^+(i\omega_n; \mathbf{r}, \mathbf{r}).\end{aligned} \tag{30}$$

This yields the gap equation,

$$\Delta_i = \sum_j g_{ij}N_j\Delta_j \int d\xi_j \frac{1}{E_j}\tanh\left(\frac{E_j}{2T}\right), \tag{31}$$

where $E_j = \sqrt{\xi_j^2 + |\Delta_j|^2}$ and T is the temperature where we set Boltzmann constant k_B to unity. N_j is the density of states at the Fermi surface. Since all the bands couple with each other through mutual interactions g_{ij}, we have one critical temperature T_c[2]. At the critical temperature $T = T_c$, this equation reads[25]

$$\Delta_i = \ln\left(\frac{2e^\gamma\omega_c}{\pi T_c}\right)\sum_j g_{ij}N_j\Delta_j, \tag{32}$$

for the cutoff energy ω_c. γ denotes the Euler constant. Here we assume the same cutoff energy in all the interactions. The system of equations in eq.(30) yields a set of differential equations[25]

$$\begin{aligned}\Delta_j^*(\mathbf{r}) &= \ln\left(\frac{2e^\gamma\omega_c}{\pi T}\right)\sum_\ell g_{j\ell}^* N_\ell \Delta_\ell^*(\mathbf{r}) \\ &+ \frac{7\zeta(3)}{48(\pi T_c)^2}\sum_\ell g_{j\ell}^* N_\ell v_\ell^2\left(\nabla + i\frac{2e}{\hbar c}\mathbf{A}\right)^2\Delta_\ell^*(\mathbf{r}) \\ &- \frac{7\zeta(3)}{8(\pi T_c)^2}\sum_\ell g_{j\ell}^* N_\ell \Delta_\ell^*(\mathbf{r})|\Delta_\ell(\mathbf{r})|^2.\end{aligned} \tag{33}$$

Here, e is the charge of the electron, and v_ℓ is the electron velocity at the Fermi surface in the ℓ-th band. The Planck constant $\hbar$ has been dropped except in front of the vector potential $\mathbf{A}$. We set $a_{mn} = g_{mn}N_n$. Then, the equation for Δ_j is

$$0 = \left[a_{jj}\ln\left(\frac{2e^\gamma\omega_c}{\pi T}\right) - 1\right]\Delta_j + \ln\left(\frac{2e^\gamma\omega_c}{\pi T}\right)\sum_{\ell(\neq j)} a_{j\ell}\Delta_\ell + \cdots. \tag{34}$$

In the second term of the right-hand side we can replace T by T_c near the transition temperature. We define the matrix $A = (a_{ij})$ and its inverse $A^{-1} = (b_{j\ell})$. Then the equation for Δ_j reads

$$0 = \left[a_{jj}\ln\left(\frac{2e^\gamma\omega_c}{\pi T}\right) - 1\right]\Delta_j + \sum_{\ell(\neq j)} a_{j\ell}\sum_m b_{\ell m}\Delta_m + \cdots. \tag{35}$$

For example, for $j = 1$ we obtain

$$\begin{aligned}0 = g_{11}\Big[&\left(N_1\ln\left(\frac{2e^\gamma\omega_c}{\pi T}\right) - \frac{1}{\det G}(G^{-1})_{11}\right)\Delta_1 \\ &- \frac{1}{\det G}(G^{-1})_{12}\Delta_2 - \frac{1}{\det G}(G^{-1})_{13}\Delta_3\Big] + \cdots,\end{aligned} \tag{36}$$

where $G = (g_{ij})$ is the matrix of coupling constants. To obtain the multi-band Ginzburg-Landau functional, we multiply eq.(35) by $\Delta_j^* N_j$ and take a summation with respect to j. We use the gap equation $\Delta_\ell^* = \eta\sum_j g_{\ell j}^* N_j\Delta_j^*$ at $T = T_c$. Then the energy functional density f is

$$\begin{aligned}f &= -\sum_j\left(N_j\ln\frac{2e^\gamma\omega_c}{\pi T} - (G^{-1})_{jj}\right)|\Delta_j|^2 \\ &+ \sum_{j\ell}\Delta_j^*(G^{-1})_{j\ell}\Delta_\ell \\ &- \frac{7\zeta(3)}{48\pi^2T_c^2}\sum_\ell N_\ell v_\ell^2\Delta_\ell^*\left(\nabla - i\frac{2e}{\hbar c}\mathbf{A}\right)^2\Delta_\ell \\ &+ \frac{7\zeta(3)}{16\pi^2T_c^2}\sum_\ell N_\ell|\Delta_\ell|^4.\end{aligned} \tag{37}$$

Here we neglected unimportant constants. The fourth order term is simply given by $(|\Delta_\ell|^2)^2$. We can also derive this functional using the functional integral method.

IV Unconventional isotope effect in multi-band superconductors

In this section we discuss the isotope effect in multi-band superconductors. The iron-based high-temperature superconductors have attracted extensive researches since the discovery of the oxypnictide $LaFeAsO_{1-x}F_x$[5]. Soon after this discovery, superconductivity has been found in several related compounds such as $BaFe_2As_2$[26], LiFeAs[27, 28] and $Fe_{1+x}Se$[29, 30]. There are numerous experimental studies regarding the electronic states of the new family of iron-based superconductor[31–36]. The undoped samples exhibit the antiferromagnetic transition[33, 34], and show the superconducting transition with electron doping[5]. The band structure calculations indicate that the Fermi surfaces are composed of two hole-like cylinders around Γ, a three-dimensional Fermi surface, and two electron-like cylinder around M for LaFeAsO[37].

Regarding the pairing symmetry of these iron-based superconductors, several experiments such as the penetration depth measurements[38] and ARPES[39, 40] indicate fully opened gaps around the Fermi surfaces. In contrast, the NMR relaxation rate $1/T_1$ shows the T^3 law, suggesting a nodal gap state such as a d-wave pairing state[34, 41–43]. There is a theoretical proposal that the $\pm s$-wave state (denoted as $s_\pm$-wave state) is most promising in the iron-based superconductors[44]. The isotope effect gives a key issue in understanding the symmetry and mechanism of superconductivity. Recently, the inverse isotope effect $\alpha \approx -0.18$ has been reported[7] for (Ba,K)Fe_2As_2 superconductor with a transition temperature $T_c \sim 38$K, where the isotope exponent α is defined as $T_c \sim M^{-\alpha}$ for the isotropic mass M. This surprising observation of the inverse isotope effect provides us important information for the pairing mechanism. The normal isotope effect was also reported, showing $\alpha \approx 0.37$[45]. Two reports will suggest completely different mechanism of superconductivity and guideline to seek new high-T_c superconductors.

We show a formula of the isotope effect on the basis of a two-band model[8, 46]. We examine a simplified model with two bands called α and β, where α-band and β-band represent conduction bands with the hole-like and electron-like Fermi surfaces, respectively. The pairing interactions in iron-based systems are classified into two categories; one is the interaction mediated by the electron-phonon interaction and the other one is mediated by the magnetic interactions. We also note that there are intra-band electron-phonon and inter-band electron-phonon interactions, which are denoted as $V_{ph}^{\alpha\alpha}$ and $V_{ph}^{\alpha\beta}$, respectively. This is also the case for the magnetic interaction and we denote the intra-band and inter-band interactions as $V_{AF}^{\alpha\alpha}$ and $V_{AF}^{\alpha\beta}$, respectively. The effective pairing interaction $V_{AF}^{\mu\rho}$ ($\mu,\rho = \alpha$ or β) is in general brought about by the perturbation in terms of the intra-orbital Coulomb interaction U, the inter-orbital Coulomb interaction U', the Hund coupling J and the pair-transfer integral (exchange-like integral) K[48–50]. Up to the second order in U, the main contributions come from the magnetic susceptibility

$$\chi_{\mu\rho}(\mathbf{q}) = \frac{1}{N}\sum_{\mathbf{k}} \frac{f(\xi_{\mathbf{k}+\mathbf{q}\mu}) - f(\xi_{\mathbf{k}\rho})}{\xi_{\mathbf{k}\rho} - \xi_{\mathbf{k}+\mathbf{q}\mu}}, \tag{38}$$

where $\xi_{\mathbf{k}\mu}$ is the dispersion of the μ band for $\mu = \alpha$ and β. The good nesting quality between α and β pockets will give rise to a superconducting state with the opposite signs with each other in these pockets.

Since the ratio $2\Delta/(k_BT_c)$ is near the BCS value[51, 52], we apply the weak-coupling BCS procedure to evaluate the critical temperature. We have two characteristic energies denoted as ω_{AF} and ω_{ph}, where we simply assume that $\omega_{ph} = \omega_{ph}^{\alpha\alpha} = \omega_{ph}^{\alpha\beta}$, that is, we have the same intra-band and inter-band phonon cut-off energies. The coupled gap equations are

$$\Delta^\alpha(\mathbf{k}) = -\frac{1}{N}\sum_{\mathbf{k}'}\sum_{i=ph,AF}\sum_{\mu=\alpha,\beta} V_i^{\alpha\mu}(\mathbf{k},\mathbf{k}')\frac{\Delta^\mu(\mathbf{k}')}{2E_{\mathbf{k}'}^\mu} \times \tanh\left(\frac{E_{\mathbf{k}'}^\mu}{2k_BT}\right), \tag{39}$$

and that for Δ^β. Here, $E_{\mathbf{k}}^\mu = \sqrt{\Delta^\mu(\mathbf{k})^2 + \xi_{\mathbf{k}}^2}$. The energy range of the pairing interaction $V_{ph}^{\mu\nu}(\mathbf{k},\mathbf{k}')$ is $0 \le |\xi_k| \le \omega_{ph}$ and $0 \le |\xi_{k'}| \le \omega_{ph}$, and that of $V_{AF}^{\mu\nu}(\mathbf{k},\mathbf{k}')$ is $0 \le |\xi_k| \le \omega_{AF}$ and $0 \le |\xi_{k'}| \le \omega_{AF}$, for μ, $\nu = \alpha$, β. We assume that $\omega_{ph} < \omega_{AF}$. Outside of these ranges they vanish. Then, the gap equations are written as

$$\Delta^\alpha(\mathbf{k}) = -\sum_{\mu=\alpha,\beta} N^\mu(0)\Big[\int_{-\omega_{ph}}^{\omega_{ph}} d\xi_{k'} V_{ph}^{\alpha\mu}(\mathbf{k},\mathbf{k}') + \int_{-\omega_{AF}}^{\omega_{AF}} d\xi_{k'} V_{AF}^{\alpha\mu}(\mathbf{k},\mathbf{k}')\Big]\frac{\Delta^\mu(\mathbf{k}')}{2E_{\mathbf{k}'}^\mu}\tanh\left(\frac{E_{\mathbf{k}'}^\mu}{2k_BT}\right), \tag{40}$$

and that for Δ^β, where $N^\mu(0)$ is the density of states at the Fermi level.

We define the coupling constants $\lambda_i^{\alpha\alpha}$ and $\lambda_i^{\alpha\beta}$ (i=ph, AF):

$$\lambda_{AF}^{\mu\nu} = \langle N^\nu(0)V_{AF}^{\mu\nu}(\mathbf{k},\mathbf{k}')\rangle_{FS}, \tag{41}$$

$$\lambda_{ph}^{\mu\nu} = -\langle N^\nu(0)V_{ph}^{\mu\nu}(\mathbf{k},\mathbf{k}')\rangle_{FS}, \tag{42}$$

for μ, $\nu = \alpha$, β. To obtain a self-consistent solution to gap equations, we set $\Delta^\mu(\mathbf{k}) = \Delta_1^\mu$ for $0 \le |\xi_k| \le \omega_{ph}$ and $\Delta^\mu(\mathbf{k}) = \Delta_2^\mu$ for $\omega_{ph} < |\xi_k| \le \omega_{AF}$, for $\mu = \alpha$, β. Then, the gap equations for T_c are

$$\begin{aligned}\Delta_1^\alpha &= (\lambda_{ph}^{\alpha\alpha} - \lambda_{AF}^{\alpha\alpha})\Delta_1^\alpha \ln\left(\frac{2e^\gamma\omega_{ph}}{\pi k_BT_c}\right) - \lambda_{AF}^{\alpha\alpha}\Delta_2^\alpha\ln\frac{\omega_{AF}}{\omega_{ph}} \\ &+ (\lambda_{ph}^{\alpha\beta} - \lambda_{AF}^{\alpha\beta})\Delta_1^\beta \ln\left(\frac{2e^\gamma\omega_{ph}}{\pi k_BT_c}\right) - \lambda_{AF}^{\alpha\beta}\Delta_2^\beta\ln\frac{\omega_{AF}}{\omega_{ph}},\end{aligned} \tag{43}$$

$$\begin{aligned}\Delta_2^\alpha &= -\lambda_{AF}^{\alpha\alpha}\Delta_1^\alpha \ln\left(\frac{2e^\gamma\omega_{ph}}{\pi k_BT_c}\right) - \lambda_{AF}^{\alpha\alpha}\Delta_2^\alpha\ln\frac{\omega_{AF}}{\omega_{ph}} \\ &- \lambda_{AF}^{\alpha\beta}\Delta_1^\beta \ln\left(\frac{2e^\gamma\omega_{ph}}{\pi k_BT_c}\right) - \lambda_{AF}^{\alpha\beta}\Delta_2^\beta\ln\frac{\omega_{AF}}{\omega_{ph}},\end{aligned} \tag{44}$$

and those for Δ_1^β and Δ_2^β. We set $\lambda_{ph}^{\alpha\beta} = \lambda_{ph}^{\beta\alpha}$ and $\lambda_{AF}^{\alpha\beta} = \lambda_{AF}^{\beta\alpha}$ since the mutual pair transfer interactions are the same between bands α and β. For simplicity, we assume that $\lambda_{ph}^{\alpha\alpha} = \lambda_{ph}^{\beta\beta}$, $\lambda_{AF}^{\alpha\alpha} = \lambda_{AF}^{\beta\beta}$ and $N^\alpha = N^\beta$.

Let us first consider the solution to this coupled equation, with the $s_\pm$ symmetry satisfying $\Delta_i^\alpha = -\Delta_i^\beta$ ($i = 1$, 2). From eq.(44), the ratio $y \equiv \Delta_2^\alpha/\Delta_1^\alpha = \Delta_2^\beta/\Delta_1^\beta$ is written as

$$y = \frac{\lambda_{AF}^{\alpha\beta} - \lambda_{AF}^{\alpha\alpha}}{1 + (\lambda_{AF}^{\alpha\alpha} - \lambda_{AF}^{\alpha\beta})\ln(\omega_{AF}/\omega_{ph})}\ln\left(\frac{2e^\gamma\omega_{ph}}{\pi k_BT_c}\right). \tag{45}$$

We obtain the critical temperature, by substituting y into eq.(43),

$$k_BT_c = \frac{2e^\gamma}{\pi}\omega_{ph}\exp\left(-\frac{1}{\lambda_{ph} + \lambda_{AF}^*}\right), \tag{46}$$

where $\lambda_{ph}=\lambda_{ph}^{\alpha\alpha}-\lambda_{ph}^{\alpha\beta}$, $\lambda_{AF}=\lambda_{AF}^{\alpha\beta}-\lambda_{AF}^{\alpha\alpha}$, and $\lambda_{AF}^*=\lambda_{AF}/(1-\lambda_{AF}\ln(\omega_{AF}/\omega_{ph}))$. It is obvious that we cannot obtain a consistent solution if we assume that $y=1$, that is, Δ^μ are constant. The above derivation of T_c is very simple and natural in the BCS approximation, and thus we can discuss the isotope effect on the basis of this formula[7]. The isotope coefficient α is derived as

$$\alpha=\frac{1}{2}\Big[1-\left(\frac{\lambda_{AF}^*}{\lambda_{ph}^{\alpha\alpha}-\lambda_{ph}^{\alpha\beta}+\lambda_{AF}^*}\right)^2\Big]. \tag{47}$$

The physics that leads to negative α is very clear. In the pairing state with $s_\pm$ symmetry, the negative $\alpha<0$ occurs if the inter-band electron-phonon coupling $\lambda_{ph}^{\alpha\beta}$ is larger than the intra-band one $\lambda_{ph}^{\alpha\alpha}$. Thus, the inverse isotope effect stems from the inter-band electron-phonon interaction. For the experimental value $\alpha\approx-0.18$, we have

$$\lambda_{ph}/\lambda_{AF}^*\approx-0.14. \tag{48}$$

Thus the contribution of the electron-phonon interaction is about 10 to 20 percent of that of the magnetic interaction. If $\lambda_{ph}>0$, we of course obtain the normal isotope effect $\alpha>0$. To obtain $\alpha\approx 0.4$, we must have $\lambda_{ph}/\lambda_{AF}^*\approx 1.24$, i.e., λ_{ph} is greater than λ_{AF}^*. The large α obviously indicates that the attractive interaction mainly originates from the electron-phonon interaction. In this case the role of λ_{AF}^* is to determine the sign of the gap function in each band. If the large isotope effects are partly caused by pair breaking due to impurities for the gap function with opposite signs in two bands, the real values of α may be reduced, for instance, $\alpha\sim 0.2$. In this case it would be possible to explain consistently the negative and positive α with varying the coupling constant of the inter-band electron-phonon coupling $\lambda_{ph}^{\alpha\beta}$. If we set $\lambda_{ph}=-0.1$ and $\omega_{ph}=200$K, we obtain $\lambda_{AF}^*=0.714$ and $T_c\sim 40$K for $\alpha=-0.18$. For $\lambda_{ph}=-0.1$, we can set, for example, $\lambda_{ph}^{\alpha\alpha}=0.1$ and $\lambda_{ph}^{\alpha\beta}=0.2$ or $\lambda_{ph}^{\alpha\alpha}=0.2$ and $\lambda_{ph}^{\alpha\beta}=0.3$, which are of the order of the values reported by a density functional calculation[47]. If we neglect the inter-band electron-phonon coupling $\lambda_{ph}^{\alpha\beta}$, we have the positive α. We do not need large λ_{ph} to get small or moderate values of positive α. For example, $\lambda_{ph}=0.1$ and 0.2 with $\lambda_{AF}^*=0.714$ result in $\alpha=0.115$ and 0.195, respectively.

Second, let us investigate the s_{++} state. In this case, we adopt $\Delta_i^\alpha=\Delta_i^\beta$ (i=1,2). We obtain, from eq.(44),

$$y=-\frac{\lambda_{AF}^{\alpha\alpha}+\lambda_{AF}^{\alpha\beta}}{1+(\lambda_{AF}^{\alpha\alpha}+\lambda_{AF}^{\alpha\beta})\ln(\omega_{AF}/\omega_{ph})}\ln\left(\frac{2e^\gamma\omega_{ph}}{\pi k_BT_c}\right). \tag{49}$$

The substitution of y to eq.(43) yields

$$k_BT_c=\frac{2e^\gamma\omega_{ph}}{\pi}\exp\left(-\frac{1}{\lambda_{ph}^+-(\lambda_{AF}^+)^*}\right), \tag{50}$$

where $\lambda_{ph}^+=\lambda_{ph}^{\alpha\alpha}+\lambda_{ph}^{\alpha\beta}$, $\lambda_{AF}^+=\lambda_{AF}^{\alpha\alpha}+\lambda_{AF}^{\alpha\beta}$ and $(\lambda_{AF}^+)^*=\lambda_{AF}^+/(1+\lambda_{AF}^+\ln(\omega_{AF}/\omega_{ph}))$. Since $d\ln(k_BT_c)/d\ln\omega_{ph}=1-[(\lambda_{AF}^+)^*/(\lambda_{ph}^+-(\lambda_{AF}^+)^*)]^2$, the isotope coefficient is

$$\alpha=\frac{1}{2}\Big[1-\left(\frac{(\lambda_{AF}^+)^*}{\lambda_{ph}^+-(\lambda_{AF}^+)^*}\right)^2\Big]. \tag{51}$$

This gives the positive isotope effect $\alpha>0$, except the case where $(\lambda_{AF}^+)^*<\lambda_{ph}^+<2(\lambda_{AF}^+)^*$. Hence, the isotope effect is probably normal in the s_{++}-pairing state.

V Half-quantum Vortex in a two-band superconductor

In this section we consider a half-quantum flux vortex in a two-band superconductor[10, 11] and discuss an analogy with a monopole. We write the order parameters as

$$\psi_j=\rho_je^{i\theta_j}, \tag{52}$$

where $\rho_j=|\psi_j|$ is a real quantity. For simplicity, we assume that the coefficients of the Josephson terms are real: $\gamma_{ij}=\gamma_{ji}^*=\gamma_{ji}$. The free energy density is denoted as f, that is, the free energy is given by the integral of f over the space. f is written as

$$\begin{aligned} f &= \sum_j\alpha_j\rho_j^2+\frac{1}{2}\sum_j\beta_j\rho_j^4-2\gamma_{12}\rho_1\rho_2\cos(\theta_1-\theta_2)\\ &- \sum_jK_j\rho_je^{-i\theta_j}\left(\nabla+i\frac{2\pi}{\phi_0}\mathbf{A}\right)^2(\rho_je^{i\theta_j})+\frac{1}{8\pi}\mathbf{H}^2. \end{aligned} \tag{53}$$

Here ϕ_0 is a quantum of flux: $\phi_0=hc/(2|e|)$. We focus on the role of phases of the order parameters; we assume that

$$\rho_j=\rho,\quad K_j=K, \tag{54}$$

and define new phase variables

$$\Phi=\theta_1+\theta_2,\quad \phi=\theta_1-\theta_2. \tag{55}$$

The free energy density is

$$\begin{aligned} f &= 2\alpha\rho^2+\beta\rho^4-2\rho^2\gamma_{12}\cos(\phi)\\ &- 2K\rho\nabla^2\rho+2K\frac{4\pi^2}{\phi_0^2}\rho^2\mathbf{A}^2+\frac{1}{2}K\rho^2(\nabla\Phi)^2\\ &+ K\frac{4\pi}{\phi_0}\rho^2\mathbf{A}\cdot\nabla\Phi+\frac{1}{2}K\rho^2(\nabla\phi)^2+\frac{1}{8\pi}\mathbf{H}^2 \end{aligned} \tag{56}$$

where $\alpha=(1/2)\sum_j\alpha_j$ and $\beta=(1/2)\sum_j\beta_j$.

Let us focus on the phase difference ϕ. Excitations that are connected to this variable is sometimes called Leggett mode[9]. The equation of motion for ϕ reads

$$2\gamma_{12}\rho^2\sin\phi-K\rho^2\nabla^2\phi=0. \tag{57}$$

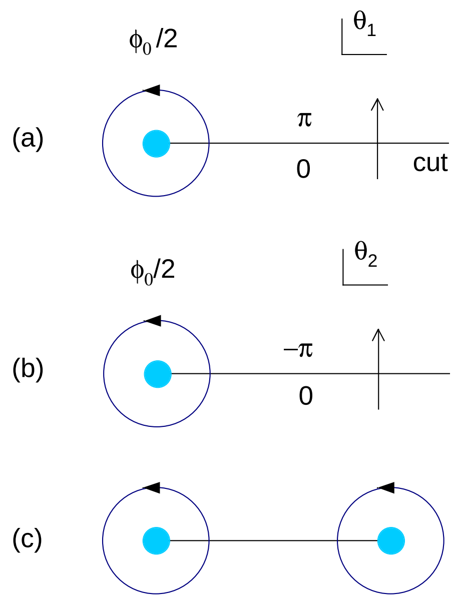

FIG. 1: Half-quantum flux vortex with line singularity. The phase variables θ_1 and θ_2 have line singularities, as shown in (a) and (b). A pair of two half-flux vortices connected by the singularity (domain wall) is shown in (c).

We define $\alpha = 2\gamma_{12}/K$ and adopt that α is positive. The sign of α does not matter because we can change the sign of $\sin\phi$ by shifting the variable ϕ. We consider a one-dimensional-like solution where ϕ has spatial dependence only in one direction, for example, in x direction. In this case the equation for ϕ is

$$\frac{d^2\phi}{dx^2} = \alpha \sin\phi. \tag{58}$$

We use the boundary condition such that $\phi \to 0$ as $x \to -\infty$ and $\phi \to 2\pi$ as $x \to \infty$. Then we have a kink solution:

$$\phi = \pi + 2\sin^{-1}(\tanh(\sqrt{\alpha}x)). \tag{59}$$

The phase difference ϕ changes from 0 to 2π across the kink. This means that θ_1 changes from 0 to π and at the same time θ_2 changes from 0 to $-\pi$. In this case, a half-quantum-flux vortex exists at the end of the kink. This is shown in Fig.2 where the half-quantum vortex is at the edge of the cut (kink). A net change of θ_1 is 2π by a counterclockwise encirclement of the vortex, and that of θ_2 vanishes. Then, we have a half-quantum flux vortex.

The half-quantum vortex can be interpreted as a monopole. Let us assume that there is a cut, namely, kink on the real axis for $x > 0$. The phase θ_1 is represented by

$$\theta_1 = \frac{1}{2}\text{Im}\log\zeta, \tag{60}$$

where

$$\zeta = x + iy. \tag{61}$$

The singularity of θ_j can be transferred to a singularity of the gauge field by a gauge transformation. Let us consider the fictitious z axis perpendicular to the x-y plane. The gauge potential (1-form) is given by

$$A_\pm = -\frac{1}{2}\frac{1}{r(z\pm r)}(ydx - xdy) = \frac{1}{2}(\pm 1 - \cos\theta)d\varphi, \tag{62}$$

where $r = \sqrt{x^2+y^2+z^2}$, and θ and φ are Euler angles. $A_\pm$ correspond to the gauge potential in the upper and lower hemisphere $H_\pm$, respectively. $A_\pm$ are connected by

$$A_+ = A_- + d\varphi. \tag{63}$$

This is the $U(1)$ bundle P over the sphere S^2. At $z = 0$ A_+ coincides with the gauge field A for half-quantum vortex. The Chern class is defined as

$$c_1(P) = -\frac{1}{2}F = -\frac{1}{2}dA_+. \tag{64}$$

The Chern number is given as

$$\begin{aligned} C_1 &= \int_{S^2} c_1 = -\frac{1}{2\pi}\int_{S^2} F \\ &= -\frac{1}{2\pi}\Big[\int_{H_+} dA_+ \int_{H_-} dA_-\Big] = 1. \end{aligned} \tag{65}$$

VI Fractional Quantum Flux Vortices

A Three-Band Superconductor

In this section let us discuss 1/3-quantized flux vorties in three-band superconductors. The Ginzburg-Landau free energy for a three-band superconductor is

$$\begin{aligned} f &= \sum_j \alpha_j\rho_j^2 + \frac{1}{2}\sum_j \beta_j\rho_j^4 - 2\gamma_{12}\rho_1\rho_2\cos(\theta_1-\theta_2) \\ &- 2\gamma_{23}\rho_2\rho_3\cos(\theta_2-\theta_3) - 2\gamma_{31}\rho_3\rho_1\cos(\theta_3-\theta_1) \\ &- \sum_j K_j\rho_j e^{-i\theta_j}\left(\nabla + i\frac{2\pi}{\phi_0}\mathbf{A}\right)^2(\rho_j e^{i\theta_j}) + \frac{1}{8\pi}\mathbf{H}^2. \end{aligned} \tag{66}$$

Here the order parameters are written as

$$\psi_j = \rho_j e^{i\theta_j}, \tag{67}$$

where $\rho_j = |\psi_j|$ is a real quantity. Since we focus on the role of phases of the order parameters, we assume that

$$\rho_j = \rho, \quad K_j = K, \tag{68}$$

and define new phase variables

$$\Phi = \theta_1+\theta_2+\theta_3, \quad \varphi_1 = \theta_1-\theta_2, \quad \varphi_2 = \theta_2-\theta_3. \tag{69}$$

Then the free energy density is

$$\begin{aligned} f &= 3\alpha\rho^2 + \frac{3}{2}\beta\rho^4 - 2\rho^2[\gamma_{12}\cos(\varphi_1) + \gamma_{23}\cos(\varphi_2) \\ &+ \gamma_{31}\cos(\varphi_1+\varphi_2)] - 3K\rho\nabla^2\rho + 3K\frac{4\pi^2}{\phi_0^2}\rho^2\mathbf{A}^2 \\ &+ \frac{1}{3}K\rho^2(\nabla\Phi)^2 + K\frac{4\pi}{\phi_0}\rho^2\mathbf{A}\cdot\nabla\Phi + \frac{1}{8\pi}\mathbf{H}^2 \\ &+ \frac{1}{3}K\rho^2[(\nabla\varphi_1)^2 + (\nabla\varphi_2)^2 + (\nabla(\varphi_1+\varphi_2))^2], \quad (70) \end{aligned}$$

where $\alpha = (1/3)\sum_j \alpha_j$ and $\beta = (1/3)\sum_j \beta_j$. The stationary conditions with respect to φ_j lead to

$$\gamma_{12}\sin\varphi_1 + \gamma_{31}\sin(\varphi_1+\varphi_2) - \frac{1}{3}K[\nabla^2\varphi_1 + \nabla^2(\varphi_1+\varphi_2)] = 0, \quad (71)$$

$$\gamma_{23}\sin\varphi_1 + \gamma_{31}\sin(\varphi_1+\varphi_2) - \frac{1}{3}K[\nabla^2\varphi_2 + \nabla^2(\varphi_1+\varphi_2)] = 0. \quad (72)$$

B Phase Potential and Chiral States

In this section we examine the ground state of the system with the potential

$$\begin{aligned} V = &-2\gamma_{12}\rho_1\rho_2\cos(\varphi_1) - 2\gamma_{23}\rho_2\rho_3\cos(\varphi_2) \\ &- 2\gamma_{31}\rho_3\rho_1\cos(\varphi_1+\varphi_2). \end{aligned} \quad (73)$$

If we set $\varphi_3 = \theta_3 - \theta_1$, we have $\varphi_1 + \varphi_2 + \varphi_3 = 0$ (mod2π). The minimum of this potential is dependent on the signs of the coefficients $\gamma_{ij}\rho_i\rho_j$ of the Josephson terms. We define $\Gamma_1 = -2\gamma_{12}\rho_1\rho_2$, $\Gamma_2 = -2\gamma_{23}\rho_2\rho_3$, and $\Gamma_3 = -2\gamma_{31}\rho_3\rho_1$. The potential is written as

$$V = \Gamma_1\cos(\varphi_1) + \Gamma_2\cos(\varphi_2) + \Gamma_3\cos(\varphi_3). \quad (74)$$

We assume that the absolute values $|\Gamma_i|$ are almost equal in magnitude. Then there are four cases to be examined as shown in Table I. When all the Γ_i are negative, we have the minimum at $\varphi_1 = \varphi_2 = \varphi_3 = 0$ (Case I). If we change the sign of Γ_3, this produces a frustration effect and φ_i take fractional values. For example, when all the $|\Gamma_i|$ are equal, we have a minimum at (φ_1,φ_2,φ_3)=($\pi/3$,$\pi/3$,$4\pi/3$). In this state the order parameters are complex and thus the time reversal symmetry is broken. The case IV also exhibits a similar state with fractional values of φ_i. If all the $|\Gamma_i|$ are the same, the ground state is at (φ_1,φ_2,φ_3)=($2\pi/3$,$2\pi/3$,$2\pi/3$) (Fig.2).

C Double Sine-Gordon Equation and Kinks

We consider the solution for $\gamma_{12} = \gamma_{23}$. In this case we have a solution with $\varphi_1 = \varphi_2 \equiv \varphi$. The variable φ satisfies the double sine-Gordon equation,

$$K\nabla^2\varphi - \gamma_{12}\sin\varphi - \gamma_{31}\sin(2\varphi) = 0. \quad (75)$$

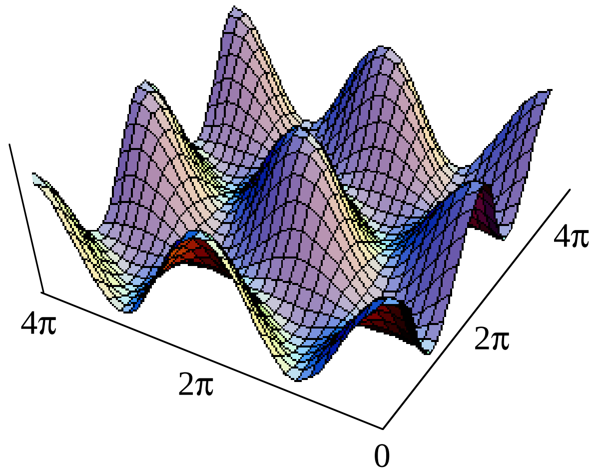

FIG. 2: Contour map of V for $\Gamma_1 = \Gamma_2 = \Gamma_3 > 0$. Black and white dots indicate minima of the potential V. Dotted line is the path in the valley connecting two minima.

TABLE II: Classification of the ground state of the potential V of the Josephson interactions. φ_i in the cases III and IV are for $|\Gamma_1| = |\Gamma_2| = |\Gamma_3|$.

	Γ_1	Γ_2	Γ_3	φ_1	φ_2	φ_3	
Case I	−	−	−	0	0	0	
Case II	+	+	−	π	π	0	
Case III	−	−	+	$\pi/3$	$\pi/3$	$4\pi/3$	chiral state
Case IV	+	+	+	$2\pi/3$	$2\pi/3$	$2\pi/3$	chiral state

Let us consider the kink solution of the one-dimensional double sine-Gordon equation. The energy functional is

$$E = \int \left[\frac{1}{2}K_0\left(\frac{d\varphi}{dx}\right)^2 + V(\varphi)\right]dx, \quad (76)$$

where $K_0 = 2K\rho^2$ and the potential V is

$$V(\varphi) = V_0\left(\cos\varphi + \frac{u}{2}\cos(2\varphi)\right). \quad (77)$$

We defined $V_0 = -\gamma_{12}\rho^2$ and $u = \gamma_{31}/\gamma_{12}$. There are two cases to be examined: (1) $\gamma_{12} < 0$ and (2) $\gamma_{12} > 0$. We show the classification of the double sine-Gordon model in Table II.

First consider the case (1) $V_0 > 0$. The potential $V(\varphi)$ has a minimum at $\varphi = \varphi_0 \equiv \cos^{-1}(-1/(2u))$ if $u > 1/2$:

$$V(\varphi_0) = V_0\left(-\frac{1}{4u} - \frac{u}{2}\right). \quad (78)$$

For $u \le 1/2$, we have a minimum at $\varphi = \pi$:

$$V(\pi) = V_0 \left(-1 + \frac{u}{2}\right). \tag{79}$$

In the case $u > 1/2$ we have a chiral state at $\varphi = \varphi_0$ and a kink solution that travels from one minimum to the other minimum. The double sine-Gordon equation

$$\frac{d^2\varphi}{dx^2} = -\frac{V_0}{K_0}(\sin\varphi + u\sin(2\varphi)), \tag{80}$$

can be integrated for the boundary conditions $d\varphi/dx \to 0$ and $\varphi \to \varphi_+$ as $x \to \infty$ and $d\varphi/dx \to 0$ and $\varphi \to -\varphi_+$ as $x \to -\infty$. Here φ_+ is a solution of $\cos(\varphi) = -1/(2u)$ in the range $-\pi < \varphi < \pi$. Since the equation above is integrated as

$$\left(\frac{d\varphi}{dx}\right)^2 = \frac{2V_0}{K_0}\left(\cos\varphi + \frac{u}{2}\cos(2\varphi) + \frac{1+2u^2}{4u}\right), \tag{81}$$

we obtain the kink solution as The kink structure is shown in Fig.4.

In the case $V_0 > 0$ and $u \le 1/2$, there are minima at $\varphi = \pi(\mathrm{mod} 2\pi)$ in the potential V. Thus, we have a 2π-kink solution in this case. The boundary conditions should be $\varphi \to \pi$ as $x \to \infty$ and $\varphi \to -\pi$ as $x \to -\infty$, or vice versa. Since we obtain

$$\left(\frac{d\varphi}{dx}\right)^2 = 4\frac{V_0}{K_0}(1-2u)\cos^2\left(\frac{\varphi}{2}\right)\left(1 + \frac{2u}{1-2u}\cos^2\left(\frac{\varphi}{2}\right)\right), \tag{82}$$

the kink solution is given by

$$\varphi(x) = \cos^{-1}\left(1 - \frac{2\sinh^2(rx)}{\cosh^2(rx) - 2u}\right), \tag{83}$$

where $r = \sqrt{V_0(1-2u)/K_0}$.

Second, let us consider the case (2) $V_0 < 0$. There are minima at $\varphi = 0$ (mod2π) for $u > -1/2$, and thus the 2π kink solution exists. We obtain

$$\varphi(x) = \cos^{-1}\left(\frac{2\sinh^2(sx)}{\cosh^2(sx) + 2u} - 1\right), \tag{84}$$

where $s = \sqrt{|V_0|(1+2u)/K_0}$. The kinks in this case are presented in Fig.5. For large u, the kink shows a characteristic at $x = 0$ because the potential has a local minimum at $\varphi = 0$ for $u > 1/2$. We have a possibility to find some specific features in the excited state due to this anomaly. For $u < -1/2$ we have a fractional-π kink that is given by

$$\varphi(x) = \tan^{-1}\left(\frac{1+2|u|t(x)}{\sqrt{4u^2-1}}\right) - \tan^{-1}\left(\frac{1+2|u|/t(x)}{\sqrt{4u^2-1}}\right), \tag{85}$$

where

$$t(x) = \exp\left(\sqrt{2\frac{|V_0|}{K_0}|u|\left(1 - \frac{1}{4u^2}\right)}x\right). \tag{86}$$

For $u = -1$, this chiral solution satisfies the boundary condition that $\varphi \to -\pi/3$ as $x \to -\infty$ and $\varphi \to \pi/3$ as $x \to \infty$ as shown in Fig.6.

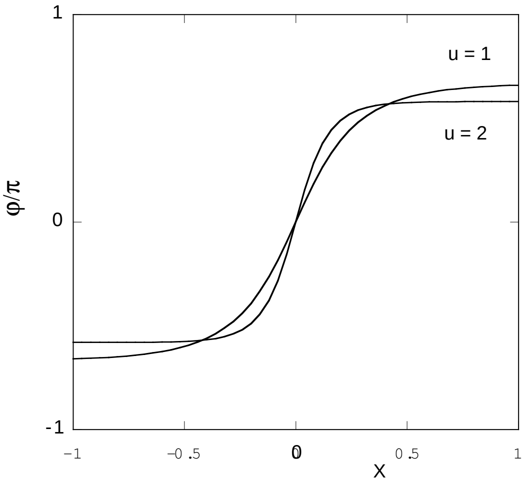

FIG. 3: φ as a function of $X = \sqrt{2V_0/K_0}x$ for $V_0 > 0$ and $u = 1$, 2. Fractional kink structure is shown. For $u = 1$, $\varphi(-\infty) = -2\pi/3$ and $\varphi(\infty) = 2\pi/3$ which we call the $4\pi/3$ kink.

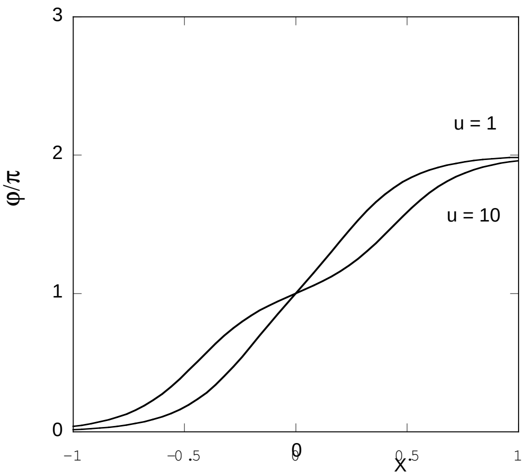

FIG. 4: φ as a function of $X = \sqrt{(u+1/2)(2|V_0|/K_0)}x$ for $V_0 < 0$ and $u = 1$ and 10. This shows 2π-kink structure. For large u, the kink shows a saddle-like structure at $x = 0$. This is because the potential V has a local minimum at $\varphi = \pi$ for $u > 1/2$.

TABLE III: Classification of the double sine-Gordon model with the potential $V(\varphi) = V_0(\cos(\varphi) + (u/2)\cos(2\varphi))$. $V(\varphi)$ has minima at $\varphi = \varphi_0$ (mod2π).

V_0	u	φ_0	kink
$V_0 > 0$	$u > 1/2$	$\cos^{-1}(-1/(2u))$	fractional-π kink chiral
$V_0 > 0$	$u < 1/2$	π	2π-kink
$V_0 < 0$	$u > -1/2$	0	2π-kink
$V_0 < 0$	$u < -1/2$	$\cos^{-1}(-1/(2u))$	fractional-π kink chiral

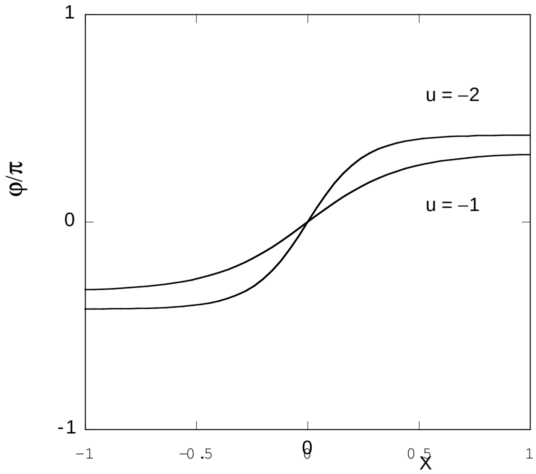

FIG. 5: φ as a function of $X = \sqrt{2|V_0|/K_0}x$ for $V_0 < 0$ and $u = -1$ and -2. This shows 2π-kink structure. For $u = -1$, $\varphi(-\infty) = -\pi/3$ and $\varphi(\infty) = \pi/3$.

D Energy of Kinks

We call the double sine-Gordon model the chiral sine-Gordon model when the conditions $V_0 > 0$ and $u > 1/2$ or $V_0 < 0$ and $u < -1/2$ hold. The energy functional for the one-dimensional system $-L/2 < x < L/2$ is

$$E = \int_{-L/2}^{L/2} \Big[\frac{1}{2}K_0\left(\frac{d\varphi}{dx}\right)^2 + V(\varphi)\Big]dx. \tag{87}$$

We assume that L is sufficiently large. If φ satisfies the stationary condition $\delta E/\delta\varphi = 0$, we obtain $K_0 d^2\varphi/dx^2 = dV(\varphi)/d\varphi$. From this we have

$$\frac{1}{2}K_0\left(\frac{d\varphi}{dx}\right)^2 = V(\varphi) - C_0, \tag{88}$$

where $C_0 = V(\varphi_0)$ with the condition that $d\varphi/dx \to 0$ as $\varphi \to \varphi_0$. Then the energy is

$$E = C_0 L + \sqrt{2K_0}\int_{\varphi_-}^{\varphi_+} \sqrt{V(\varphi) - C_0}\,d\varphi, \tag{89}$$

where $\varphi_+ = \varphi(L/2)$ and $\varphi_- = \varphi(-L/2)$, and both satisfy $d\varphi/dx \to 0$ as $\varphi \to \varphi_\pm$. For $V_0 > 0$ and $u > 1/2$, we obtain the energy of fractional-π kink state as

$$E_{f-kink} = 2\sqrt{2K_0V_0u}\Big[\sqrt{1-\frac{1}{4u^2}} + \frac{1}{2u}\cos^{-1}\left(-\frac{1}{2u}\right)\Big] + C_0 L \tag{90}$$

where $C_0 = V(\varphi_0) = V(\varphi_+) = V(\varphi_-) = -V_0(1/(4u) + u/2)$. This coincides with the energy obtained by substituting the kink solution directly to the energy functional.

The energy of the 2π-kink for $V_0 < 0$ and $u > 0$ is

$$\begin{aligned} E_{2\pi-kink} &= C_1 L + 4\sqrt{K_0V_0}\Big[\sqrt{1+2u} \\ &- \frac{1}{2\sqrt{2u}}\log\Big|\left(1-\frac{2u}{1+2u}\right)/\left(1+\frac{2u}{1+2u}\right)\Big|\Big], \end{aligned} \tag{91}$$

where $C_1 = -|V_0|(1+u/2)$.

Let us consider the case $V_0 > 0$ and $u > 1/2$. By using the inequality $a^2 + b^2 \geq 2|ab|$ for real a and b, we obtain

$$\begin{aligned} E &= \int\Big[\frac{1}{2}K_0\left(\frac{d\varphi}{dx}\right)^2 + V_0\left(\cos\varphi + \frac{u}{2}\cos(2\varphi)\right)\Big]dx \\ &= C_0L + \int\Big[\frac{1}{2}K_0\left(\frac{d\varphi}{dx}\right)^2 + V_0u\left(\cos\varphi + \frac{1}{2u}\right)^2\Big]dx \\ &\geq C_0L + 2\sqrt{\frac{1}{2}K_0V_0u}\int\Big|\frac{d\varphi}{dx}\left(\cos\varphi + \frac{1}{2u}\right)\Big|dx \\ &= \sqrt{2K_0V_0u}\Big[\sin\varphi_+ - \sin\varphi_- + \frac{1}{2u}(\varphi_+ - \varphi_-)\Big] + C_0L, \end{aligned} \tag{92}$$

for the one-kink solution that satisfies $d\varphi/dx \geq 0$ and $\cos\varphi + 1/(2u) \geq 0$. In the case of one fractional-π kink shown above, we obtain

$$E \geq \sqrt{2K_0V_0u}\Big[2\sqrt{1-\frac{1}{4u^2}} + \frac{1}{u}\cos^{-1}\left(-\frac{1}{2u}\right)\Big] + C_0L. \tag{93}$$

where we adopt that $0 \leq \cos^{-1}\left(-\frac{1}{2u}\right) \leq \pi$. The lower bound of the energy coincides with the energy in eq.(90). Here, we define the conserved current

$$J^\mu = \frac{1}{2A}\epsilon^{\mu\nu}\partial_\nu\varphi, \tag{94}$$

with the charge

$$Q = \int_{-\infty}^{\infty} J^0(x)dx = \frac{1}{2A}[\varphi(x=\infty) - \varphi(x=-\infty)], \tag{95}$$

where $\epsilon^{\mu\nu}$ is the antisymmetric symbol and $x^0 = t$, $x^1 = x$. A is the normalization constant defined by $A = \cos^{-1}(-1/(2u))$ with the value in the range $0 \leq A \leq \pi$. $\partial_\mu J^\mu = 0$ follows immediately from the antisymmetric symbol $\epsilon^{\mu\nu}$ and does not depend on the equation of motion. The kink has $Q = 1$, and an antikink with $Q = -1$ exists that has a configuration with $\varphi(-\infty) = A$ and $\varphi(\infty) = -A$. If we assume that $\cos\varphi + 1/(2u) > 0$ for the kink solution, the energy inequality is written as

$$E \geq \text{sign}Q\sqrt{2K_0V_0u}(\sin\varphi_+ - \sin\varphi_-) + \sqrt{\frac{2K_0V_0}{u}}A|Q| + C_0L. \tag{96}$$

This is an inequality of Bogomol'nyi type. For the kink satisfying $\cos\varphi + 1/(2u) < 0$, the inequality is

$$\begin{aligned} E \geq &-\Big[\text{sign}Q\sqrt{2K_0V_0u}(\sin\varphi_+ - \sin\varphi_-) + \sqrt{\frac{2K_0V_0}{u}}A|Q|\Big] \\ &+ C_0L. \end{aligned} \tag{97}$$

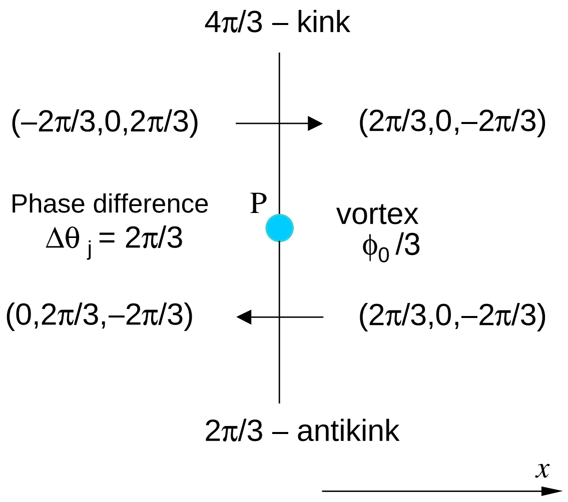

FIG. 6: Kink, antikink and a fractional flux vortex for $\Gamma_1 = \Gamma_2 = \Gamma_3 > 0$. The vortex is at the point P with flux $\phi_0/3$ where ϕ_0 is the flux quantum. We start from $(\theta_1, \theta_2, \theta_3) = (-2\pi/3, 0, 2\pi/3)$ to reach $(0, 2\pi/3, -2\pi/3)$ (modulo 2π) through the $4\pi/3$-kink and $2\pi/3$-antikink. $\varphi_1 = \theta_1 - \theta_2$ goes from $-2\pi/3$ to $2\pi/3$ crossing the $4\pi/3$-kink, and φ_1 goes from $2\pi/3$ to $4\pi/3 \equiv -2\pi/3$ (mod 2π) through the $2\pi/3$-kink.

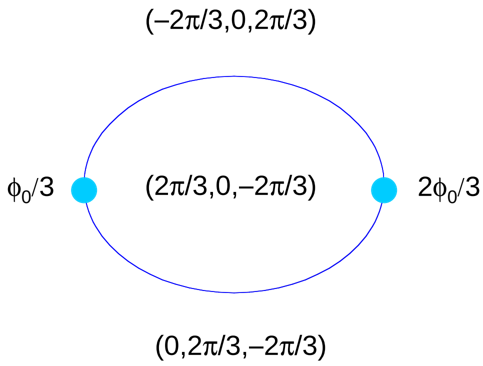

FIG. 7: Two-vortex bound state with line singularities in the time-reversal symmetry broken state. The phase variables θ_i $(i = 1, 2, 3)$ have singularities that are described by kinks in Fig.4. The total flux is ϕ_0. Topologically, the flux $2\phi_0/3$ is equivalent to $-\phi_0/3$. Thus, this state corresponds to the meson under the duality transformation between charge and magnetic flux.

VII Fractional Vortices and Bound States

In general, there are solutions of vortices with fractional quantum flux in multi-band superconductors. Kinks in the space of phase variables θ_j play a central role for the existence of fractional flux vortices as in the case of half-quantum flux vortices.

In three-band superconductors, the fractional-flux vortex exists in the chiral case as well as the non-chiral case. Since we have the fractional-π kink in the chiral state (cases III and IV), the new types of vortices with fractional flux quanta exist on a domain wall of the kink[13–15]. The kink considered in the previous section is a one-dimensional structure in superconductors. There are many types of kinks connecting two minima of the potential in three-band superconductors.

Let us discuss the fractional vortices in the three-band model here. Suppose that two kinks, one is a kink and the other is an antikink, intersect at a point P as shown in Fig.7 in a two-dimensional xy-plane. If a vortex exists along the z axis just at the point P, the vortex should have a fractional flux quantum so that the phase change around the point P is 2π. For $V_0 > 0$ and $u > 1/2$, this is shown schematically in Fig.7. We set the phases of the order parameters $(\theta_1, \theta_2, \theta_3) = (-2\pi/3, 0, 2\pi/3)$ in some region. After crossing the $4\pi/3$ kink, they become $(2\pi/3, 0, -2\pi/3)$ where the phase variables φ_1 and φ_2 change from $-2\pi/3$ to $2\pi/3$. If there is also a domain wall of an antikink that starts from the point P as in Fig.7, we have the phases $(0, 2\pi/3, -2\pi/3)$ after we cross the antikink. Here, we obtain the phase difference between the initial and final states (see Fig.7). In this case, the vortex that is located through the point P along the z axis should have a fractional flux quantum $\phi_0/3$. Thus, in the chiral region of three-band superconductors, the existence of fractional vortices is easily concluded in this way.

In the three-band model, the fractional vortex has two line singularities (kinks) in the phases of the gap function as shown in Fig.7. From Fig.7, we have a two-vortex bound state as presented in Fig.8 in the chiral state. Two vortices form a 'molecule' by two kinks. This state may have lower energy than the vortex state with quantum flux ϕ_0 since the magnetic energy $(5/9)\phi_0^2$ is smaller than ϕ_0^2 of the unit flux. The energy of kinks is proportional to the distance R between two fractional vortices if R is large. Thus, the attractive interaction works between them if R is sufficiently large.

Three-vortex bound states are also formulated: they are shown in Figs.9, 10 and 11. The first two figures indicate bound states in the time-reversal symmetry broken state. The last one is for the unbroken state[53]. These states correspond to baryons if we regard the magnetic flux as charge.

In quantum Chromodynamics (QCD), quarks do not appear as asymptotic states. Quarks are constitutes of nucleons and mesons and cannot be separated arbitrary far from the rest of the constituents. This is the hypothesis of confinement[57]. The interquark potential is expected to be a linearly increasing function so that quarks

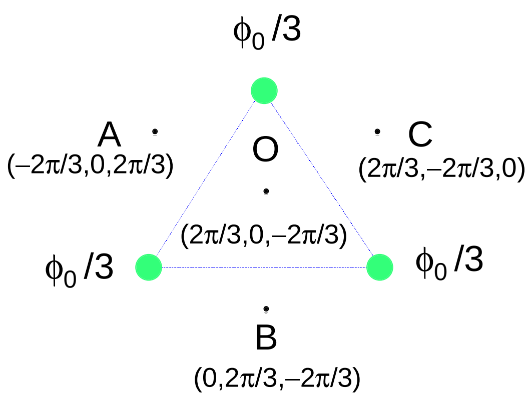

FIG. 8: Three-vortex bound state with line singularities in the time-reversal symmetry broken state. Each vortex has $\phi_0/3$ and the total flux is ϕ_0. The phase variables θ_i $(i = 1, 2, 3)$ have singularities that are fractional-π kinks. One can read one $\phi_0/3$ as $-2\phi_0/3$. Hence, this state corresponds to the neutron.

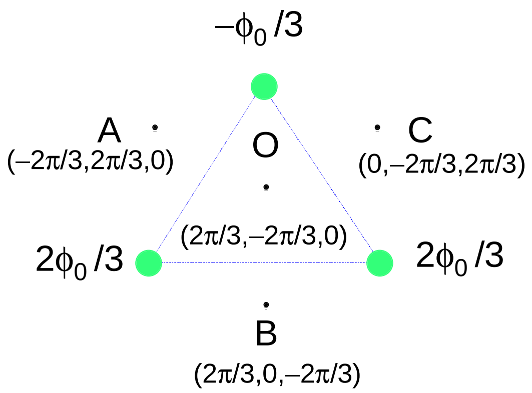

FIG. 9: Three-vortex bound state with line singularities in the time-reversal symmetry broken state. This state corresponds to the proton.

are confined. To understand such a potential, one may consider a thin flux tube between separated quarks. The energy of the flux tube is proportional to the length of the tube. This is an analogy to magnetic vortices in type-II superconductors where magnetic fields penetrate into superconductors with quantized flux. Because the energy of vortex per unit length is constant, the total energy of vortex is proportional to its length. The confinement of quarks can be understood as confinement in dual superconductor model as a result of the dual to Meissner

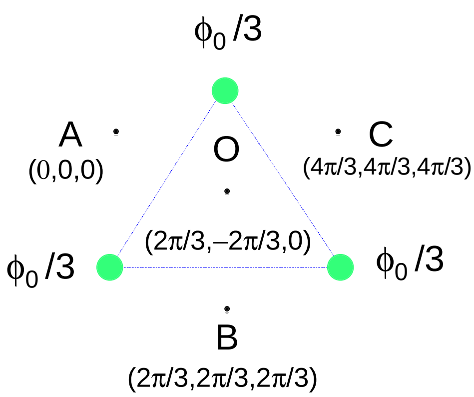

FIG. 10: Three-vortex bound state with line singularities in the time-reversal symmetric state. Each vortex has $\phi_0/3$ and the total flux is ϕ_0. In this state, the region including the point O has higher energy.

effect.

Let us consider a pair of half-quantum flux vortices discussed in section V. Because two half-flux vortices are connected by a line of singularity (namely, domain wall), the potential energy between them is proportional to the separation of two vortices,

$$V(r) = \sigma r, \tag{98}$$

for large r where σ is a constant proportional to $\sqrt{K_0 V_0}$. This has an analogy to quarks in a charged pion because of a linear confinement potential. The linear potential in a superconductor originates from the kink in the phase space of the order parameters. The kink of half-quantum flux vortex is a defect in two-dimensional space and this attracts two separated vortices. When two half-quantum vortices become separated far away each other, it is more energetically favorable for a new half-quantum pair to appear, than to let the kink to extend further.

A vortex with unit flux ϕ_0 has the energy proportional to ϕ_0^2, and a half-quantum flux has the magnetic energy which is proportional to $\phi_0^2/4$. Hence, the separation L between two half-flux vortices is determined by the energy balance,

$$\frac{1}{2}\phi_0^2 = AL, \tag{99}$$

for some constant A. Because $A \propto \sqrt{V_0}$ and V_0 is proportional to the strength of Josephson coupling γ, L increases as the Josephson coupling γ decreases.

Goryo et al proposed a mechanism of deconfinement of two vortices due to an entropy effect[58]. They calculated the entropy and estimated the temperature above which two fractional vortices are deconfined. The obtained deconfinement temperature T_{dec} is very close to T_c, and thus free fractional vortices hardly exist at $T < T_c$.

VIII Leggett mode and gauge field

Let us consider the Ginzburg-Landau free energy density of a two-band superconductor without the Josephson term in a magnetic field:

$$\begin{aligned} f &= -(\alpha_1|\psi_1|^2 + \alpha_2|\psi_2|^2) + \frac{1}{2}(\beta_1|\psi_1|^4 + \beta_2|\psi_2|^4) \\ &+ \frac{\hbar^2}{2m_1}|\left(\nabla - ie^*\mathbf{A}\right)\psi_1|^2 + \frac{\hbar^2}{2m_2}|\left(\nabla - ie^*\mathbf{A}\right)\psi_2|^2 \\ &+ \frac{1}{8\pi}(\nabla\times\mathbf{A})^2, \end{aligned} \tag{100}$$

where ψ_j ($j = 1, 2$) are the order parameters and $e^* = 2e$. We set $\hbar = c = 1$. This functional is not invariant under the transformation:

$$\psi_j \to \exp(i\theta_j)\psi_j, \quad \mathbf{A} \to \mathbf{A} + \frac{1}{e^*}\nabla\chi. \tag{101}$$

The functional is not invariant for any choice of χ. This means that the free energy functional of a two-band superconductor is not gauge invariant even without the Josephson term[59]. Let us adopt that the phase of ψ_j is θ_j: $\psi_j = e^{i\theta_j}|\psi_j|$, and define $\Phi = \theta_1 + \theta_2$ and $\varphi = \theta_1 - \theta_2$. We use the relation,

$$\begin{aligned} \nabla(e^{i\theta_1}\psi_1) &= (i\nabla\theta_1)e^{i\theta_1}\psi_1 + e^{i\theta_1}\nabla\psi_1 \\ &= \left(i\frac{1}{2}\nabla\Phi + i\frac{1}{2}\nabla\varphi\right)e^{i\theta_1}\psi_1 + e^{i\theta_1}\nabla\psi_1 \end{aligned} \tag{102}$$

and that for ψ_2. Then the free energy is

$$\begin{aligned} f &= -(\alpha_1|\psi_1|^2 + \alpha_2|\psi_2|^2) + \frac{1}{2}(\beta_1|\psi_1|^4 + \beta_2|\psi_2|^4) \\ &+ \frac{\hbar^2}{2m_1}\left|\left(\nabla - ie^*\mathbf{A} - ie^*\mathbf{B}\right)|\psi_1|\right|^2 \\ &+ \frac{\hbar^2}{2m_2}\left|\left(\nabla - ie^*\mathbf{A} + ie^*\mathbf{B}\right)|\psi_2|\right|^2 + \frac{1}{8\pi}(\nabla\times\mathbf{A})^2, \end{aligned} \tag{103}$$

where the field $\mathbf{B}$ is the derivative of the phase difference φ,

$$\mathbf{B} = \frac{1}{2e^*}\nabla\varphi. \tag{104}$$

Here we consider a generalized model where the real quantity $|\psi_j|$ is replaced by a complex field ψ_j. This means that we consider a model that is a generalization of the model of superconductors:

$$\begin{aligned} f_c &= -(\alpha_1|\psi_1|^2 + \alpha_2|\psi_2|^2) + \frac{1}{2}(\beta_1|\psi_1|^4 + \beta_2|\psi_2|^4) \\ &+ \frac{\hbar^2}{2m_1}\left|\left(\nabla - ie^*\mathbf{A} - ie^*\mathbf{B}\right)\psi_1\right|^2 \\ &+ \frac{\hbar^2}{2m_2}\left|\left(\nabla - ie^*\mathbf{A} + ie^*\mathbf{B}\right)\psi_2\right|^2 + \frac{1}{8\pi}(\nabla\times\mathbf{A})^2, \end{aligned} \tag{105}$$

f_c is invariant when the fields $\mathbf{A}$ and $\mathbf{B}$ are transformed as

$$\psi_j \to \exp(i\bar{\theta}_j)\psi_j, \quad \mathbf{A} \to \mathbf{A} + \frac{1}{2e^*}\nabla\bar{\Phi}, \quad \mathbf{B} \to \mathbf{B} + \frac{1}{2e^*}\nabla\bar{\varphi}, \tag{106}$$

where $\bar{\Phi}$ and $\bar{\varphi}$ are defined similarly for $\bar{\theta}_j$. Let us examine the properties of this model. When $m_1 = m_2 = m$, the kinetic part becomes

$$f_{ckin} = \frac{\hbar^2}{2m}\left|\left(\nabla - ie^*\mathbf{A}\sigma_0 - ie^*\mathbf{B}\sigma_3\right)\psi\right|^2 + \frac{1}{8\pi}(\nabla\times\mathbf{A})^2, \tag{107}$$

where

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{108}$$

σ_0 is unit matrix and σ_3 is the Pauli matrix. This is a part of $SU(2)\times U(1)$ gauge theory (Weinberg-Salam model). The gauge field $\mathbf{A}$ appears as $\mathbf{A}-\mathbf{B}$ and $\mathbf{A}+\mathbf{B}$, or (after the gauge transformation) $\mathbf{A}$ and $\mathbf{A} + 2\mathbf{B} \equiv \mathbf{Z}$. Thus, the Laggett fluctuation mode $\mathbf{B}$ appears as a linear combination with $\mathbf{A}$. The masses of gauge bosons are

$$m_A = \frac{\hbar}{c}\frac{1}{\lambda_2} \propto |\psi_2|, \quad m_Z = \frac{\hbar}{c}\frac{1}{\lambda_1} \propto |\psi_1|, \tag{109}$$

where λ_1 and λ_2 are

$$\lambda_1 = \sqrt{\frac{1}{4\pi}\left(\frac{c}{e^*}\right)^2\frac{m_1}{\rho_1^2}}, \quad \lambda_2 = \sqrt{\frac{1}{4\pi}\left(\frac{c}{e^*}\right)^2\frac{m_2}{\rho_2^2}}, \tag{110}$$

with $\rho_j \equiv |\psi_j|$. The penetration depth λ_L is given by

$$\frac{1}{\lambda_L^2} = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}. \tag{111}$$

This is because the current is given as (from the variational condition $\delta f/\delta\mathbf{A} = 0$)

$$\begin{aligned} \mathbf{j} &= \frac{1}{2}\hbar e^*\left(\frac{\rho_1^2}{m_1} + \frac{\rho_2^2}{m_2}\right)\nabla\Phi + \frac{1}{2}\hbar e^*\left(\frac{\rho_1^2}{m_1} - \frac{\hbar}{m_2}\right)\nabla\varphi \\ &- \frac{(e^*)^2}{c}\left(\frac{\rho_1^2}{m_1} + \frac{\rho_2^2}{m_2}\right)\mathbf{A}. \end{aligned} \tag{112}$$

This leads to

$$\mathrm{rot}\mathbf{j} = -\frac{(e^*)^2}{c}\left(\frac{\rho_1^2}{m_1} + \frac{\rho_2^2}{m_2}\right)\mathrm{rot}\mathbf{A}. \tag{113}$$

Since $\mathrm{rot}(\nabla\varphi) = 0$, the Leggett mode gives no contribution to the penetration depth.

If we add the Josephson term to the free energy,

$$\gamma(\psi_1^\dagger\psi_2 + \psi_2^\dagger\psi_1) = 2\gamma|\psi_1\psi_2|\cos(\varphi), \tag{114}$$

the gauge invariance is broken.

It is straightforward to generalize the free energy to an N-band superconductor. For the three-band case, the covariant derivative reads

$$D_\mu = \partial_\mu - ie^*A_\mu - ie^*\sqrt{\frac{3}{2}}B_\mu^8\lambda_8 - ie^*\sqrt{\frac{3}{2}}B_\mu^3\lambda_3, \tag{115}$$

where λ_8 and λ_3 are diagonal Gell-Mann matrices. There are 2 Leggett modes in the three-band case. In general, in the N-band case, we have $N-1$ Leggett modes. $N-1$ equals the rank of $SU(N)$. The rank is the number of elements of Cartan subalgebra. Let $t_1, \cdots, t_{N-1}$ be elements of the Cartan subalgebra of $SU(N)$. Then, the covariant derivative is

$$D_\mu = \partial_\mu - ie^*A_\mu - ie^*\sum_{j=1}^{N-1} B_\mu^j t_j, \tag{116}$$

and the kinetic part of the free energy density is given by

$$f_{ckin} = \frac{\hbar^2}{2m}|D_\mu\psi|^2 + \frac{1}{8\pi}(\nabla\times\mathbf{A})^2. \tag{117}$$

Here, $\psi = (\psi_1, \cdots, \psi_N)^t$ is a scalar field of order parameters.

The Leggett mode is none other than the gauge field in the generalized model. The Leggett modes are represented by the diagonal part of the gauge field. Let us write the gauge field B_μ in the form

$$B_\mu = \sum_{j=1}^{N-1} B_\mu^j t_j + \sum_{a=1}^{N^2-N} C_\mu^a X_a. \tag{118}$$

B_μ^j $(j = 1, \cdots, N-1)$ denote the diagonal elements of the vector field and C_μ^a are the off-diagonal elements of the vector field. The Leggett modes correspond to the diagonal part, and this is the abelian projection of $SU(N)$ by 'tHooft[20]. A singularity of the gauge field B_μ^j appears as a monopole. This singularity leads to a half-quantum flux vortex in a two-band superconductor. Fractionally quantized vortices arise as a result of singularities of the gauge field.

IX Summary

Internal quantum phases play an important and interesting role in multi-band superconductors. The Nambu-Goldstone mode, which appears as a result of the spontaneous breaking of overall phase invariance, becomes a plasma mode with the energy gap and is usually not important in superconductors. The phase difference Leggett mode instead can play an important role with the small or vanishing energy gap.

In iron-based superconductors the unconventional isotope effect appears when the signs of two gap functions are opposite to each other. This is clearly a multi-band effect. The phase difference φ satisfies the sine-Gordon equation. A kink solution to the sine-Gordon equation leads to the existence of a half-quantum flux vortex in a magnetic field. In a three-band superconductor, there is a possibility of chiral superconducting state with time-reversal symmetry breaking. There appears further a new state in a four-band superconductor. A massless mode surely appears in the superconducting gap when the four bands are equivalent. The phase difference mode is represented as a gauge field and the action is gauge invariant if we neglect the pari-transfer term (Josephson term). The Josephson term breaks the gauge invariance. There is an attractive analogy between the theory of multi-band superconductors and the gauge theory. The action of the N-band superconductor is given by the ablian projection of an $SU(N)$ gauge theory.

Analogies between ideas in particle physics and corresponding ones physics of superconductivity are also interesting. A half-quantum flux vortex can be regarded as a monopole, and fractionally quantized-flux vortices bear a striking resemblance to quarks. Two half-quantum vortices form a pair with a linearly increasing potential, which corresponds to mesons. They are not allowed to become separated far away each other for $T < T_c$. In a three-band superconductor, three fractional-flux vortices can form a bound state with the total flux ϕ_0 or 0, which corresponds to the proton or neutron, respectively, after the dual transformation between charge and magnetic flux.

Heisenberg considered a microscopic theory of ferromagnets using quantum mechanics, and later employed an analogy with ferromagnets to construct a unified theory of elementary particles[60, 61]. Nambu imported the idea of spontaneous symmetry breaking in condensed-matter physics to particle theory to account for the mass of fermions, especially nucleons. He first noticed the resemblance between the Dirac equation in field theory and the gap equation in superconductivity theory.

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