Chapter 4

Renormalization Group Theory of Effective Field Theory Models in Low Dimensions

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Additional information is available at the end of the chapter

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Abstract

We discuss the renormalization group approach to fundamental field theoretic models in low dimensions. We consider the models that are universal and frequently appear in physics, both in high-energy physics and condensed matter physics. They are the non-linear sigma model, the ϕ^4 model and the sine-Gordon model. We use the dimensional regularization method to regularize the divergence and derive renormalization group equations called the beta functions. The dimensional method is described in detail.

Keywords: renormalization group theory, dimensional regularization, scalar model, non-linear sigma model, sine-Gordon model

1. Introduction

The renormalization group is a fundamental and powerful tool to investigate the property of quantum systems [1–15]. The physics of a many-body system is sometimes captured by the analysis of an effective field theory model [16–19]. Typically, effective field theory models are the ϕ^4 model, the non-linear sigma model and the sine-Gordon model. Each of these models represents universality as a representative of a universal class.

The ϕ^4 model is the model of a phase transition, which is often referred to as the Ginzburg-Landau model. The renormalization of the ϕ^4 model gives a prototype of renormalization group procedures in field theory [20–24].

The non-linear sigma model appears in various fields of physics [15, 25–27] and is the effective model of Quantum chromodynamics (QCD) [28] and also that of magnets (ferromagnetic and anti-ferromagnetic materials) [29–32]. This model exhibits an important property called the asymptotic freedom. The non-linear sigma model is generalized to a model with fields that take values in a compact Lie group G [33–42]. This is called the chiral model.

The sine-Gordon model also has universality [43–49]. The two-dimensional (2D) sine-Gordon model describes the Kosterlitz-Thouless transition of the 2D classical XY model [50, 51]. The 2D sine-Gordon model is mapped to the Coulomb gas model where particles interact with each other through a logarithmic interaction. The Kondo problem [52, 53] also belongs to the same universality class where the scaling equations are just given by those for the 2D sine-Gordon model, i.e. the equations for the Kosterlitz-Thouless transition [53–57]. The one-dimensional Hubbard model is also mapped onto the 2D sine-Gordon model on the basis of a bosonization method [58, 59]. The Hubbard model is an important model of strongly correlated electrons [60–65]. The Nambu-Goldstone (NG) modes in a multi-gap superconductor become massive due to the cosine potential, and thus the dynamical property of the NG mode can be understood by using the sine-Gordon model [66–71]. The sine-Gordon model will play an important role in layered high-temperature superconductors because the Josephson plasma oscillation is analysed on the basis of this model [72–75].

In this paper, we discuss the renormalization group theory for the ϕ4 theory, the non-linear sigma model and the sine-Gordon model. We use the dimensional regularization procedure to regularize the divergence [76].

2. ϕ^4 model

2.1. Lagrangian

The ϕ^4 model is given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4, \tag{1}$$

where ϕ is a scalar field and g is the coupling constant. In the unit of the momentum μ, the dimension of $\mathcal{L}$ is given by d, where d is the dimension of the space-time: $[\mathcal{L}] = \mu^d$. The dimension of the field ϕ is $(d-2)/2$: $[\phi] = \mu^{(d-2)/2}$. Because $g\phi^4$ has the dimension d, the dimension of g is given by $4 - d$: $[g] = \mu^{4-d}$. Let us adopt that ϕ has N components as $\phi = (\phi_1, \phi_2, \ldots, \phi_N)$. The interaction term ϕ^4 is defined as

$$\phi^4 = \left(\sum_{i=1}^{N} \phi_i^2\right)^2. \tag{2}$$

The Green's function is defined as

$$G_i(x-y) = -i\langle 0|T\phi_i(x)\phi_i(y)|0\rangle, \tag{3}$$

where T is the time-ordering operator and $|0\rangle$ is the ground state. The Fourier transform of the Green's function is

$$G_i(p) = \int d^{\,d}x e^{ip\cdot x} G_i(x). \tag{4}$$

In the non-interacting case with $g = 0$, the Green's function is given by

$$G_i^{(0)}(p) = \frac{1}{p^2 - m^2}, \tag{5}$$

where $p^2 = (p_0)^2 - \vec{p}^{\,2}$ for $p = (p_0, \vec{p})$.

Let us consider the correction to the Green's function by means of the perturbation theory in terms of the interaction term $g\phi^4$. A diagram that appears in perturbative expansion contains, in general, L loops, I internal lines and V vertices. They are related by

$$L = I - V + 1. \tag{6}$$

There are L degrees of freedom for momentum integration. The degree of divergence D is given by

$$D = d \cdot L - 2I. \tag{7}$$

We have a logarithmic divergence when $D = 0$. Let E be the number of external lines. We obtain

$$4V = E + 2I. \tag{8}$$

Then, the degree of divergence is written as

$$D = d \cdot L - 2I = d + (d-4)V + \left(1 - \frac{d}{2}\right)E. \tag{9}$$

In four dimensions $d = 4$, the degree of divergence D is independent of the numbers of internal lines and vertices

$$D = 4 - E \tag{10}$$

When the diagram has four external lines, $E = 4$, we obtain $D = 0$ which indicates that we have a logarithmic (zero-order) divergence. This divergence can be renormalized.

Let us consider the Lagrangian with bare quantities

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{1}{2} m_0^2 \phi_0^2 - \frac{1}{4!} g_0 \phi_0^4, \tag{11}$$

where ϕ_0 denotes the bare field, g_0 denotes the bare coupling constant and m_0 is the bare mass. We introduce the renormalized field ϕ, the renormalized coupling constant g and the renormalized mass m. They are defined by

$$\phi_0 = \sqrt{Z_\phi}\phi, \tag{12}$$

$$g_0 = Z_g g, \tag{13}$$

$$m_0^2 = m^2 Z_2/Z_\phi, \tag{14}$$

where Z_ϕ, Z_g and Z_2 are renormalization constants. When we write Z_g as

$$Z_g = Z_4/Z_\phi^2, \tag{15}$$

we have $g_0 Z_\phi^2 = gZ_4$. Then, the Lagrangian is written by means of renormalized field and constants

$$\mathcal{L} = \frac{1}{2}Z_\phi(\partial_\mu\phi)^2 - \frac{1}{2}m^2 Z_2\phi^2 - \frac{1}{4!}gZ_4\phi^4. \tag{16}$$

2.2. Regularization of divergences

2.2.1. Two-point function

We use the perturbation theory in terms of the interaction $g\phi^4$. For a multi-component scalar field theory, it is convenient to express the interaction ϕ^4 as in **Figure 1**, where the dashed line indicates the coupling g. We first examine the massless case with $m \to 0$. Let us consider the renormalization of the two-point function $\Gamma^{(2)}(p) = iG(p)^{-1}$. The contributions to $\Gamma^{(2)}$ are shown in **Figure 1**. The first term indicates $p^2 Z_\phi$ and the contribution in the second term is represented by the integral

$$I = \int \frac{d^d q}{(2\pi)^d}\frac{1}{q^2 - m^2}. \tag{17}$$

Using the Euclidean co-ordinate $q_4 = -iq_0$, this integral is evaluated as

$$I = -i\frac{\Omega_d}{(2\pi)^d}m^{d-2}\frac{1}{2}\Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right), \tag{18}$$

where Ω_d is the solid angle in d dimensions. For $d > 2$, the integral I vanishes in the limit $m \to 0$. Thus, the mass remains zero in the massless case. We do not consider mass renormalization in the massless case. Let us examine the third term in **Figure 2**.

There are $4^2 \cdot 2N + 4^2 \cdot 2^2 = 32N + 64$ ways to connect lines for an N-component scalar field to form the third diagram in **Figure 2**. This is seen by noticing that this diagram is represented as a sum of two terms in **Figure 3**.

The number of ways to connect lines is $32N$ for (a) and 64 for (b). Then we have the factor from these contributions as

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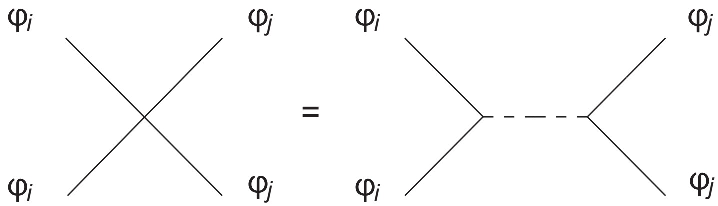

Figure 1. ϕ^4 interaction with the coupling constant g.

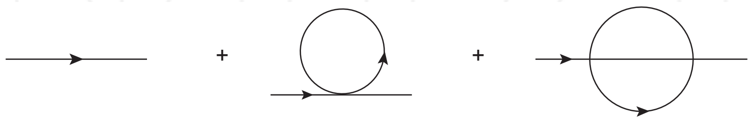

Figure 2. The contributions to the two-point function $\Gamma^{(2)}(p)$ up to the order of g^2.

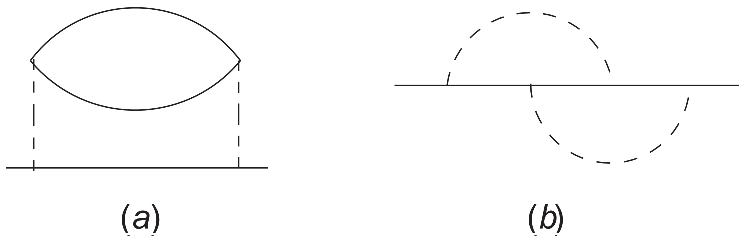

Figure 3. The third term in **Figure 2** is a sum of two configurations (a) and (b).

$$\left(\frac{1}{4!}g\right)^2(32N+64)=\frac{N+2}{18}g^2. \tag{19}$$

The momentum integral of this term is given as

$$J(k):=\int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{1}{p^2q^2(p+q+k)^2}. \tag{20}$$

The integral J exhibits a divergence in four dimensions $d = 4$. We separate the divergence as $1/\epsilon$ by adopting $d = 4 - \epsilon$. The divergent part is regularized as

$$J=-\left(\frac{1}{8\pi^2}\right)^2\frac{1}{8\epsilon}+\text{regular terms} \tag{21}$$

To obtain this, we first perform the integral with respect to q by using

$$\frac{1}{q^2(p+q+k)^2} = \int_0^1 dx \frac{1}{[q^2x+(p+q+k)^2(1-x)]^2}. \tag{22}$$

For $q' = q + (1-x)(p+k)$, we have

$$\begin{aligned}\int \frac{d^dq}{(2\pi)^d}\frac{1}{q^2(p+q+k)^2} &= \int \frac{d^dq'}{(2\pi)^d}\int_0^1 dx \frac{1}{[q'^2+x(1-x)(p+k)^2]^2} \\ &= \frac{\Omega_d}{(2\pi)^d}\int_0^1 dx\Big(x(1-x)\Big)^{\frac{d}{2}-2}\Big((p+k)^2\Big)^{\frac{d}{2}-2}\int_0^\infty dr r^{d-1}\frac{1}{(r^2+1)^2} \\ &= \frac{\Omega_d}{(2\pi)^d}\,\frac{1}{2}\,\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)^2\frac{1}{\Gamma(d-2)}\,\Big((p+k)^2\Big)^{\frac{d}{2}-2}.\end{aligned} \tag{23}$$

Here, the following parameter formula was used

$$\frac{1}{A^nB^m} = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)}\int_0^1 dx \frac{x^{n-1}(1-x)^{m-1}}{[xA+(1-x)B]^{n+m}}. \tag{24}$$

Then, we obtain

$$\begin{aligned}\int \frac{d^dp}{(2\pi)^d}\,\frac{1}{p^2\Big((p+k)^2\Big)^{2-d/2}} &= \frac{\Gamma(3-d/2)}{\Gamma(2-d/2)}\int_0^1 dx(1-x)^{1-d/2}\int \frac{d^dp'}{(2\pi)^d}\,\frac{1}{[p'^2+x(1-x)k^2]^{3-d/2}} \\ &= \frac{\Omega_d}{(2\pi)^d}\,\frac{\Gamma(3-d/2))}{\Gamma(2-d/2)}B\left(d-2,\frac{d}{2}-1\right)\frac{1}{2}B\left(\frac{d}{2},3-d\right)\,(k^2)^{d-3}.\end{aligned} \tag{25}$$

Here $B(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$. We use the formula

$$\Gamma(\epsilon) = \frac{1}{\epsilon} + \text{finite terms} \tag{26}$$

for $\epsilon \to 0$. This results in

$$\int \frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\,\frac{1}{p^2q^2(p+q+k)^2} = -\left(\frac{1}{8\pi^2}\right)^2\frac{1}{8\epsilon}\,k^2 + \text{ regular terms} \tag{27}$$

Therefore, the two-point function is evaluated as

$$\Gamma^{(2)}(p) = Z_\phi p^2 + \frac{1}{8\epsilon}\,\frac{N+2}{18}\,\left(\frac{g}{8\pi^2}\right)^2 p^2, \tag{28}$$

up to the order of $O(g^2)$. In order to cancel the divergence, we choose Z_ϕ as

$$Z_\phi = 1 - \frac{1}{8\epsilon}\,\frac{N+2}{18}\,\left(\frac{1}{8\pi^2}\right)^2 g^2. \tag{29}$$

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2.2.2. *Four-point function*

Let us turn to the renormalization of the interaction term g^4. The perturbative expansion of the four-point function is shown in **Figure 4**. The diagram (*b*) in **Figure 4**, denoted as $\Delta\Gamma_b^{(4)}$, is given by for $N = 1$:

$$\Delta\Gamma_b^{(4)}(p) = g^2 \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 - m^2)\left(((p+q)^2 - m^2\right)}. \tag{30}$$

As in the calculation of the two-point function, this is regularized as

$$\Delta\Gamma_b^{(4)}(p) = i \frac{1}{8\pi^2} \frac{1}{2\epsilon} g^2, \tag{31}$$

for $d = 4 - \epsilon$. Let us evaluate the multiplicity of this contribution for $N > 1$. For $N = 1$, we have a factor $4^2 3^2 2/4!4! = 1/2$ as shown in Eq. (30). **Figure 4c** and **d** gives the same contribution as in Eq. (31), giving the factor 3/2. For $N > 1$, there is a summation with respect to the components of ϕ. We have the multiplicity factor for the diagram in **Figure 4b** as

$$\left(\frac{1}{4!}\right)^2 2^2 2^2 2N = \frac{N}{18}. \tag{32}$$

Since we obtain the same factor for diagrams in **Figure 4c** and **d**, we have $N/6$ in total. We subtract 1/6 for $N = 1$ from 3/2 to have 8/6. Finally, the multiplicity factor is given by $(N + 8)/6$. Then, the four-point function is regularized as

$$\Delta\Gamma^{(4)}(p) = i \frac{1}{8\pi^2} \frac{N+8}{6} \frac{1}{\epsilon} g^2. \tag{33}$$

Because g has the dimension $4 - d$ such as $[g] = \mu^{4-d}$, we write g as $g\mu^{4-d}$ so that g is the dimensionless coupling constant. Now, we have

$$\Gamma^{(4)}(p) = -igZ_4\mu^\epsilon + i \frac{1}{8\pi^2} \frac{N+8}{6} \frac{1}{\epsilon} g^2. \tag{34}$$

for $d = 4 - \epsilon$ where we neglect μ^ϵ in the second term. The renormalization constant is determined as

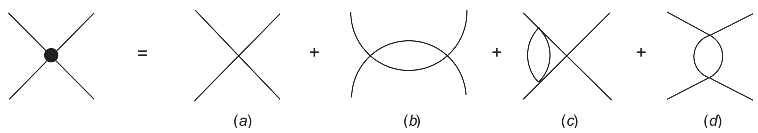

Figure 4. Diagrams for four-point function.

$$Z_4 = 1 + \frac{N+8}{6\epsilon}\frac{1}{8\pi^2}g. \tag{35}$$

As a result, the four-point function $\Gamma^{(4)}$ becomes finite.

2.3. Beta function $\beta(g)$

The bare coupling constant is written as $g_0 = Z_g g\mu^{4-d} = (Z_4/Z_\phi^2)g\mu^{4-d}$. Since g_0 is independent of the energy scale, μ, we have $\mu\partial g_0/\partial\mu = 0$. This results in

$$\mu\frac{\partial g}{\partial\mu} = (d-4)g - g\mu\frac{\partial g}{\partial\mu}\frac{\partial \ln Z_g}{\partial g}, \tag{36}$$

where $Z_g = Z_4/Z_\phi^2$. We define the beta function for g as

$$\beta(g) = \mu\frac{\partial g}{\partial\mu}, \tag{37}$$

where the derivative is evaluated under the condition that the bare g_0 is fixed. Because

$$Z_g = 1 + \frac{N+8}{6\epsilon}\frac{1}{8\pi^2}g + O(g^2), \tag{38}$$

the beta function is given as

$$\beta(g) = \frac{-\epsilon g}{1 + g\frac{\partial \ln Z_g}{\partial g}} = -\epsilon g + \frac{N+8}{6}\frac{1}{8\pi^2}g^2 + O(g^3). \tag{39}$$

$\beta(g)$ up to the order of g^2 is shown as a function of g for $d < 4$ in **Figure 5**. For $d < 4$, there is a non-trivial fixed point at

$$g_c = \epsilon\frac{48\pi^2}{N+8}. \tag{40}$$

For $d = 4$, we have only a trivial fixed point at $g = 0$.

For $d = 4$ and $N = 1$, the beta function is given by

$$\beta(g) = \frac{3}{16\pi^2}g^2 + \cdots. \tag{41}$$

In this case, the $\beta(g)$ has been calculated up to the fifth order of g [77]:

$$\beta(g) = \frac{3}{16\pi^2}g^2 - \frac{17}{3}\frac{1}{(16\pi^2)^2}g^3 + \left(\frac{145}{8} + 12\zeta(3)\right)\frac{1}{(16\pi^2)^3}g^4 + A_5\frac{1}{(16\pi^2)^4}g^5, \tag{42}$$

where

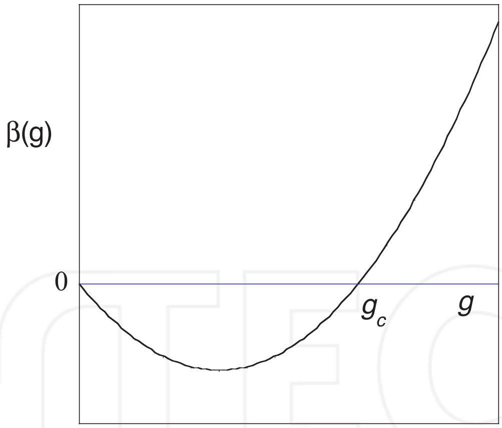

Figure 5. The beta function of g for $d < 4$. There is a finite fixed point g_c.

$$A_5 = -\left(\frac{3499}{48} + 78\zeta(3) - 18\zeta(4) + 120\zeta(5)\right), \tag{43}$$

and $\zeta(n)$ is the Riemann zeta function. The renormalization constant Z_g and the beta function $\beta(g)$ are obtained as a power series of g. We express Z_g as

$$Z_g = 1 + \frac{N+8}{6\epsilon} g + \left(\frac{b_1}{\epsilon^2} + \frac{b_2}{\epsilon}\right) g^2 + \left(\frac{c_1}{\epsilon^3} + \frac{c_2}{\epsilon^2} + \frac{c_3}{\epsilon}\right) g^3 + \cdots, \tag{44}$$

and then $\beta(g)$ is written as

$$\begin{aligned} \beta(g) &= -\epsilon g + \epsilon g^2 \left[\frac{N+8}{6\epsilon} + 2\left(\frac{b_1}{\epsilon^2} + \frac{b_2}{\epsilon}\right) g + \frac{(N+8)^2}{36\epsilon^2} g + \cdots\right] \\ &= -\epsilon g + \frac{N+8}{6} g^2 - \frac{9N+42}{36} g^3 + \cdots \end{aligned} \tag{45}$$

Here, the factor $1/8\pi^2$ is included in g. The terms of order $1/\epsilon^2$ are cancelled because of

$$b_1 = -\frac{(N+8)^2}{72}. \tag{46}$$

In general, the nth order term in $\beta(g)$ is given by $n!g^n$. The function $\beta(g)$ is expected to have the form

$$\beta(g) = -\epsilon g + \frac{N+8}{6} g^2 + \cdots + n! a^n n^b c g^n + \cdots, \tag{47}$$

where a, b and c are constants.

2.4. *n*-point function and anomalous dimension

Let us consider the n-point function $\Gamma^{(n)}$. The bare and renormalized n-point functions are denoted as $\Gamma_B^{(n)}(p_i, g_0, m_0, \mu)$ and $\Gamma_R^{(n)}(p_i, g, m, \mu)$, respectively, where p_i (i = 1,…, n) indicate momenta. The energy scale μ indicates the renormalization point. $\Gamma_R^{(n)}$ has the mass dimension $n + d - nd/2$: $[\Gamma_R^{(n)}] = \mu^{n+d-nd/2}$. These quantities are related by the renormalization constant Z_ϕ as

$$\Gamma_R^{(n)}(p_i, g, m^2, \mu) \;=\; Z_\phi^{n/2}\Gamma_B^{(n)}(p_i, g_0, m_0^2, \mu). \tag{48}$$

Here, we consider the massless case and omit the mass. Because the bare quantity $\Gamma_B^{(n)}$ is independent of μ, we have

$$\frac{d}{d\mu}\Gamma_B^{(n)} = 0. \tag{49}$$

This leads to

$$\mu\frac{d}{d\mu}\left(Z_\phi^{-n/2}\Gamma_R^{(n)}\right) = 0. \tag{50}$$

Then we obtain the equation for $\Gamma_R^{(n)}$:

$$\left(\mu\frac{\partial}{\partial\mu} + \mu\frac{\partial g}{\partial\mu}\frac{\partial}{\partial g} - \frac{n}{2}\gamma_\phi\right)\Gamma_R^{(n)}(p_i, g, \mu) = 0, \tag{51}$$

where γ_ϕ is defined as

$$\gamma_\phi \;=\; \mu\frac{\partial}{\partial\mu}\ln Z_\phi. \tag{52}$$

A general solution of the renormalization equation is written as

$$\Gamma_R^{(n)}(p_i, g, \mu) = \exp\left(\frac{n}{2}\int_{g_1}^{g}\frac{\gamma_\phi(g')}{\beta(g')}dg'\right)f^{(n)}(p_i, g, \mu), \tag{53}$$

where

$$f^{(n)}(p_i, g, \mu) = F\left(p_i, \ln\mu - \int_{g_1}^{g}\frac{1}{\beta(g')}dg'\right), \tag{54}$$

for a function F and a constant g_1. We suppose that $\beta(g)$ has a zero at $g = g_c$. Near the fixed point g_c, by approximating $\gamma_\phi(g')$ by $\gamma_\phi(g_c)$, $\Gamma_R^{(n)}$ is expressed as

$$\Gamma_R^{(n)}(p_i, g_c, \mu) = \mu^{\frac{n}{2}\gamma_\phi(g_c)} f^{(n)}(p_i, g_c, \mu). \tag{55}$$

In general, we define $\gamma(g)$ as

$$\gamma(g)\ln\mu = \int_{g_1}^{g} \frac{\gamma_\phi(g')}{\beta(g')} dg', \tag{56}$$

Then, we obtain

$$\Gamma_R^{(n)}(p_i, g, \mu) = \mu^{\frac{n}{2}\gamma(g)} f^{(n)}(p_i, g, \mu). \tag{57}$$

Under a scaling $p_i \to \rho p_i$, $\Gamma_R^{(n)}$ is expected to behave as

$$\Gamma_R^{(n)}(\rho p_i,\ g_c,\ \mu) = \rho^{n+d-nd/2} \Gamma_R^{(n)}(p_i,\ g_c,\ \mu/\rho), \tag{58}$$

because $\Gamma_R^{(n)}$ has the mass dimension $n + d - nd/2$. In fact, **Figure 4b** gives a contribution being proportional to

$$\begin{aligned} g^2(\mu^{4-d})^2 \int d^d q \frac{1}{q^2(\rho p + q)^2} &= g^2(\mu^{4-d})^2 \rho^{d-4} \int d^d q \frac{1}{q^2(p+q)^2} \\ &= \rho^{4-d} g^2 \left(\frac{\mu}{\rho}\right)^{2(4-d)} \int d^d q \frac{1}{q^2(p+q)^2}, \end{aligned} \tag{59}$$

after the scaling $p_i \to \rho p_i$ for $n = 4$. We employ Eq. (58) for $n = 2$

$$\begin{aligned} \Gamma_R^{(2)}(\rho p_i,\ g_c,\ \mu) &= \rho^2 \Gamma_R^{(2)}(p_i,\ g_c,\ \mu/\rho) = \rho^2 \left(\frac{\mu}{\rho}\right)^{\gamma} f^{(2)}(p_i,\ g_c,\ \mu/\rho) \\ &= \rho^{2-\gamma} \mu^{\gamma} f^{(2)}(p_i,\ g_c,\ \mu/\rho) = \rho^{2-\gamma} \Gamma_R^{(2)}(p_i,\ g_c,\ \mu/\rho). \end{aligned} \tag{60}$$

This indicates

$$\Gamma^{(2)}(p) \ = \ p^{2-\eta} \ = \ p^{2-\gamma} \ = \ (p^2)^{1-\gamma/2}. \tag{61}$$

Thus, the anomalous dimension η is given by $\eta = \gamma$. From the definition of $\gamma(g)$ in Eq. (56), we have

$$\gamma_\phi(g) \ = \ \gamma(g) + \beta(g) \frac{\partial \gamma(g)}{\partial g} \ln \mu. \tag{62}$$

At the fixed point $g = g_c$, this leads to

$$\eta \ = \ \gamma \ = \ \gamma(g_c) \ = \ \gamma_\phi(g_c). \tag{63}$$

The exponent η shows the fluctuation effect near the critical point.

The Green's function $G(p) = \Gamma^{(2)}(p)^{-1}$ is given by

$$G(p) \;=\; \frac{1}{p^{2-\eta}}. \tag{64}$$

The Fourier transform of $G(p)$ in d dimensions is evaluated as

$$G(r) \;=\; \int \frac{1}{p^{2-\eta}} e^{ip\cdot r} d^d p \;=\; \Omega_d \frac{1}{r^{d-2+\eta}} \frac{\pi}{2\Gamma(4-\eta-d)\sin\Big((4-\eta-d)\pi/2\Big)}. \tag{65}$$

When $4 - \eta - d$ is small near four dimensions, $G(r)$ is approximated as

$$G(r) \;\approx\; \Omega_d \frac{1}{r^{d-2+\eta}}. \tag{66}$$

The definition of γ_ϕ in Eq. (52) results in

$$\gamma_\phi(g) \;=\; \mu \frac{\partial g}{\partial \mu} \frac{\partial}{\partial g} \ln Z_\phi \;=\; \beta(g) \frac{\partial}{\partial g} \ln Z_\phi. \tag{67}$$

Up to the lowest order of g, γ_ϕ is given by

$$\begin{aligned} \gamma_\phi &= \left(-\frac{1}{8\epsilon} \frac{N+1}{9} \frac{1}{(8\pi^2)^2} g \right) \beta(g) + \; O(g^3) \\ &= \frac{N+2}{72} \frac{1}{(8\pi^2)^2} g^2 \;+\; O(g^3). \end{aligned} \tag{68}$$

At the critical point $g = g_c$, where

$$\frac{1}{8\pi^2} g_c \;=\; \frac{6 \in}{N+8}, \tag{69}$$

the anomalous dimension is given as

$$\eta = \; \gamma_\phi(g_c) = \; \frac{N+2}{2(N+8)^2} \epsilon^2 + O(\epsilon^3). \tag{70}$$

For $N = 1$ and $\epsilon = 1$, we have $\eta = 1/54$.

2.5. Mass renormalization

Let us consider the massive case $m \neq 0$. This corresponds to the case with $T > T_c$ in a phase transition. The bare mass m_0 m and renormalized mass m are related through the relation $m^2 = m_0^2 Z_\phi / Z_2$. The condition $\mu\, \partial\, m_0 / \partial \mu = 0$ leads to

$$\mu\frac{\partial \ln m}{\partial\mu}=\mu\frac{\partial}{\partial\mu}\ln\frac{Z_\phi}{Z_2}. \tag{71}$$

From Eq. (50), the equation for $\Gamma_R^{(n)}$ is

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}-\frac{n}{2}\gamma_\phi+\mu\frac{\partial}{\partial\mu}\ln\left(\frac{Z_\phi}{Z_2}\right)\cdot m^2\frac{\partial}{\partial m^2}\right]\Gamma_R^{(n)}(p_i,\ g,\ \mu,\ m^2)=0. \tag{72}$$

We define the exponent ν by

$$\frac{1}{\nu}-2=\ \mu\frac{\partial}{\partial\mu}\ln\left(\frac{Z_2}{Z_\phi}\right), \tag{73}$$

then

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}-\frac{n}{2}\gamma_\phi-\left(\frac{1}{\nu}-2\right)m^2\frac{\partial}{\partial m^2}\right]\Gamma_R^{(n)}(p_i,\ g,\ \mu,\ m^2)=0. \tag{74}$$

At the critical point $g = g_c$, we obtain

$$\left[\mu\frac{\partial}{\partial\mu}-\frac{n}{2}\eta-\zeta m^2\frac{\partial}{\partial m^2}\right]\Gamma_R^{(n)}(p_i,\ g_c,\ \mu,\ m^2)=0, \tag{75}$$

where $\gamma_\phi = \eta$ and we set

$$\zeta=\ \frac{1}{\nu}-2. \tag{76}$$

At $g = g_c$, $\Gamma_R^{(n)}$ has the form

$$\Gamma_R^{(n)}(p_i,\ g_c,\ \mu,\ m^2)=\ \mu^{\frac{n}{2}}F^{(n)}(p_i,\ \mu m^{2/\zeta}). \tag{77}$$

because this satisfies Eq. (75).

In the scaling $p_i\to\rho p_i$, we adopt

$$\Gamma_R^{(n)}(\rho p_i,\ g_c,\ \mu,\ m^2)=\ \rho^{n+d-nd/2}\Gamma_R^{(n)}(p_i,\ g_c,\ \mu/\rho,\ m^2/\rho^2). \tag{78}$$

From Eq. (77), we have

$$\Gamma_R^{(n)}(k_i,\ g_c,\ \mu,\ m^2)=\ \rho^{n+d-nd/2-n\eta/2}\mu^{\frac{n}{2}\eta}F^{(n)}\left(\rho^{-1}k_i,\ \rho^{-1}\mu(\rho^{-2}m^2)^{1/\zeta}\right), \tag{79}$$

where we put $\rho p_i=k_i$. We assume that $F^{(n)}$ depends only on $\rho{-}1k_i$. We choose ρ as

$$\rho=\ (\mu m^{2/\zeta})^{\zeta/(\zeta+2)}=\ \mu\left(\frac{m^2}{\mu^2}\right)^{1/(\zeta+2)}. \tag{80}$$

This satisfies $\rho^{-1}\mu(\rho^{-2}m^2)^{1/\varsigma}=1$ and results in

$$\Gamma_R^{(n)}(k_i,\ g_c,\ \mu,\ m^2) = \mu^{d+\frac{n}{2}(2-d-\eta)}\left(\frac{m^2}{\mu^2}\right)^{\{d+\frac{n}{2}(2-d-\eta)\}\frac{1}{\varsigma+2}}\mu^{\frac{n}{2}\eta}F^{(n)}\left(\mu^{-1}\left(\frac{m^2}{\mu^2}\right)^{-\frac{1}{\varsigma+2}}k_i\right). \tag{81}$$

We take μ as a unit by setting $\mu = 1$, so that $\Gamma_R^{(n)}$ is written as

$$\Gamma_R^{(n)}(k_i,\ g_c,\ 1,\ m^2) = \ m^{2\nu\left\{d+\frac{n}{2}(2-d+\eta)\right\}}F^{(n)}(k_i m^{-2\nu}), \tag{82}$$

because $\varsigma + 2 = 1/\nu$. We can define the correlation length ξ by

$$(m^2)^{-\nu} \ = \xi. \tag{83}$$

The two-point function is written as

$$\Gamma_R^{(2)}(k, m^2) = \ m^{2\nu(2-\eta)}F^{(2)}(km^{-2\nu}). \tag{84}$$

Now let us turn to the evaluation of ν. Since $\gamma_\phi = \mu\,\partial\ln Z_\phi/\partial\mu$, from Eq. (73) ν is given by

$$\frac{1}{\nu} = 2 + \mu\frac{\partial}{\partial\mu}\ln\left(\frac{Z_2}{Z_\phi}\right) = 2 + \beta(g)\frac{\partial}{\partial g}\ln Z_2 - \gamma_\phi(g). \tag{85}$$

The renormalization constant Z_2 is determined from the corrections to the bare mass m_0. The one-loop correction, shown in **Figure 6**, is given by

$$\Sigma(p^2) = i\frac{N+2}{6}g\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2 - m_0^2}, \tag{86}$$

where the multiplicity factor is $(8 + 4N)/4!$. This is regularized as

$$\Sigma(p^2) = \ \frac{N+2}{6}g\int\frac{d^dk}{(2\pi)^d}\frac{1}{k_E^2 + m_0^2} = -\frac{N+2}{6}g\frac{1}{8\pi^2}m_0^2\frac{1}{\epsilon}, \tag{87}$$

for $d = 4–\epsilon$. Therefore the renormalized mass is

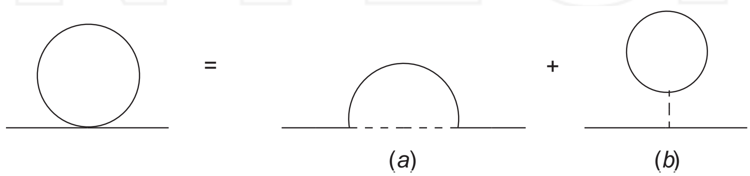

Figure 6. Corrections to the mass term. Multiplicity weights are 8 for (a) and 2N for (b).

$$m^2 = m_0^2 + \Sigma(p^2) = m_0^2\left(1 - \frac{N+2}{6\epsilon}\frac{1}{8\pi^2}g\right) \tag{88}$$

Z_2 is determined to cancel the divergence in the form m^2Z_2/Z_ϕ. The result is

$$Z_2 = 1 + \frac{N+2}{6\epsilon}\frac{1}{8\pi^2}g. \tag{89}$$

Then, we have

$$\beta(g)\frac{\partial}{\partial g}\ln Z_2 = -\frac{N+2}{6}\frac{1}{8\pi^2}g + O(g^2). \tag{90}$$

Eq. (85) is written as

$$\frac{1}{\nu} = 2 - \frac{N+2}{6}\frac{1}{8\pi^2}g_c - \eta = 2 - \frac{N+2}{N+8}\epsilon + O(\epsilon^2), \tag{91}$$

where we put $g = g_c$ and used $\eta = \gamma_\phi(g) = (N+2)/\left(2(N+8)^2\right)\cdot\epsilon$. Now the exponent ν is

$$\nu = \frac{1}{2}\left(1 + \frac{N+2}{2(N+8)}\epsilon\right) + O(\epsilon^2). \tag{92}$$

In the mean-field approximation, $\nu = 1/2$. This formula of ν contains the fluctuation effect near the critical point. For $N = 1$ and $\epsilon = 1$, we have $\nu = 1/2 + 1/12 = 7/12$.

3. Non-linear sigma model

3.1. Lagrangian

The Lagrangian of the non-linear sigma model is

$$\mathcal{L} = \frac{1}{2g}(\partial_\mu\phi)^2, \tag{93}$$

where ϕ is a real N-component field $\phi = (\phi_1,\ldots,\phi_N)$ with the constraint $\phi^2 = 1$. This model has an O(N) invariance. The field ϕ is represented as

$$\phi = (\sigma, \pi_1, \pi_2, \cdots, \pi_{N-1}) \tag{94}$$

with the condition $o^2 + \pi_1^2 + \cdots + \pi_{N-1}^2 = 1$. The fields π_i ($i = 1, \ldots, N-1$) are regarded as representing fluctuations. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2g}\{(\partial_\mu \sigma)^2 + (\partial_\mu \pi_i)^2\}, \tag{95}$$

where summation is assumed for index i. In this Section we consider the Euclidean Lagrangian from the beginning. Using the constraint $\sigma^2 + \pi_i^2 = 1$, the Lagrangian is written in the form

$$\mathcal{L} = \frac{1}{2g}(\partial_\mu \pi_i)^2 + \frac{1}{2g}\frac{1}{1-\pi_i^2}(\pi_i \partial_\mu \pi_i)^2 \tag{96}$$

$$= \frac{1}{2g}(\partial_\mu \pi_i)^2 + \frac{1}{2g}(\pi_i \partial_\mu \pi_i)^2 + \cdots \tag{97}$$

The second term in the right-hand side indicates the interaction between π_i fields. The diagram for this interaction is shown in **Figure 7**.

Here, let us check the dimension of the field and coupling constant. Since $[\mathcal{L}] = \mu^d$, we obtain $[\pi] = \mu^0$ (dimensionless) and $[g] = \mu^{2-d}$. g_0 and g are used to denote the bare coupling constant and renormalized coupling constant, respectively. The bare and renormalized fields are indicated by π_{Bi} and π_{Ri}, respectively. We define the renormalization constants Z_g and Z by

$$g_0 = g\mu^{2-d} Z_g, \tag{98}$$

$$\pi_{Bi} = \sqrt{Z}\,\pi_{Ri} \tag{99}$$

where g is the dimensionless coupling constant. Then, the Lagrangian is expressed in terms of renormalized quantities:

$$\mathcal{L} = \frac{\mu^{d-2} Z}{2g Z_g}\left\{(\partial_\mu \pi_{Ri})^2 + \frac{1}{4}(\partial_\mu \pi_{Ri}^2)^2 + \cdots\right\}. \tag{100}$$

In order to avoid the infrared divergence at d = 2, we add the Zeeman term to the Lagrangian which is written as

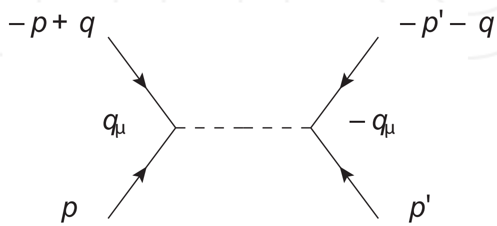

Figure 7. Lowest order interaction for π_i.

$$\mathcal{L}_Z = \frac{H_B}{g_0}\sigma = \frac{H_B}{g_0}\left(1 - \frac{Z}{2}\pi_{Ri}^2 - \frac{Z^2}{8}\pi_{Ri}^4 + \cdots\right) \tag{101}$$

$$= const. - H_B \frac{Z}{2gZ_g}\mu^{d-2}\pi_{Ri}^2 - H_B \frac{Z^2}{8gZ_g}\mu^{d-2}(\pi_{Ri}^2)^2. \tag{102}$$

Here, H_B is the bare magnetic field and the renormalized magnetic field H is defined as

$$H = \frac{\sqrt{Z}}{Z_g}H_B \tag{103}$$

Then, the Zeeman term is given by

$$\mathcal{L}_z = const. - \frac{\sqrt{Z}}{2g}H\mu^{d-2}\pi_{Ri}^2 - \frac{Z^{\frac{3}{2}}}{8g}H\mu^{d-2}(\pi_{Ri}^2)^2 + \cdots. \tag{104}$$

3.2. Two-point function

The diagrams for the two-point function $\Gamma^{(2)}(p) = G^{(2)}(p)^{-1}$ are shown in **Figure 8**. The contributions in **Figure 8c** and **d** come from the magnetic field. **Figure 8b** presents

$$I_b = \int \frac{d^dk}{(2\pi)^d}\frac{(k+p)^2}{k^2+H} = (p^2 - H)\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+H'} \tag{105}$$

where we used the formula in the dimensional regularization given as

$$\int d^dk = 0. \tag{106}$$

Near two dimensions, $d = 2 + \epsilon$, the integral is regularized as

$$I_b = (p^2 - H)\frac{\Omega_d}{(2\pi)^d}H^{\frac{d}{2}-1}\Gamma\left(\frac{d}{2}\right)\Gamma\left(1 - \frac{d}{2}\right) = -(p^2 - H)\frac{\Omega_d}{(2\pi)^d}\frac{1}{\epsilon}. \tag{107}$$

The H-term I_c in **Figure 8c** just cancels with $-H$ in I_b. The contribution I_d in **Figure 8d** has the multiplicity $2 \cdot 2 \cdot (N-1)$ because (π_i) has $N-1$ components. I_d is evaluated as

$$I_c = \frac{1}{8}\cdot 4(N-1)\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+H} = -\frac{\Omega_d}{(2\pi)^d}\frac{N-1}{2}\frac{1}{\epsilon}. \tag{108}$$

As a result, up to the one-loop-order the two-point function is

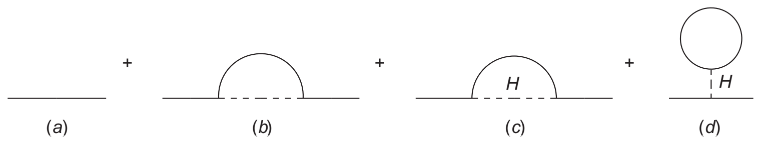

Figure 8. Diagrams for the two-point function. The diagrams (c) and (d) come from the Zeeman term.

$$\Gamma^{(2)}(p) = \frac{Z}{Z_g g} p^2 + \frac{\sqrt{Z}}{g} H - \frac{1}{\epsilon}\left(p^2 + \frac{N-1}{2} H\right), \tag{109}$$

where the factor $\Omega_d/(2\pi)^d$ is included in g for simplicity. To remove the divergence, we choose

$$\frac{Z}{Z_g} = 1 + \frac{g}{\epsilon}, \tag{110}$$

$$\sqrt{Z} = 1 + \frac{N-1}{2\epsilon} g. \tag{111}$$

This set of equations indicates

$$Z_g = 1 + \frac{N-1}{\epsilon} g + O(g^2), \tag{112}$$

$$Z = 1 + \frac{N-1}{\epsilon} g + O(g^2). \tag{113}$$

The case $N = 2$ is s special case, where we have $Z_g = 1$. This will hold even when including higher order corrections. For $N = 2$, we have one π field satisfying

$$\sigma_2 + \pi_2 = 1 \tag{114}$$

When we represent σ and π as $\sigma = \cos\theta$ and $\pi = \sin\theta$, the Lagrangian is

$$\mathcal{L} = \frac{1}{2g}\{(\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2\} = \frac{1}{2g}(\partial_\mu \theta)^2. \tag{115}$$

If we disregard the region of θ, $0 \le \theta \le 2\pi$, the field θ is a free field suggesting that $Z_g = 1$.

3.3. Renormalization group equations

The beta function $\beta(g)$ of the coupling constant g is defined by

$$\beta(g) = \mu \frac{\partial g}{\partial \mu}, \tag{116}$$

where the bare quantities are fixed in calculating the derivative. Since $\mu \partial g_0/\partial\mu = 0$, the beta function is derived as

$$\beta(g) = \frac{\epsilon g}{1 + g\frac{\partial}{\partial g}\ln Z_g} = \epsilon g - (N-2)g^2 + O(g^3), \tag{117}$$

for $d = 2 + \epsilon$. The beta function is shown in **Figure 9** as a function of g. We mention here that the coefficient $N - 2$ of g^2 term is related with the Casimir invariant of the symmetry group O(N) [34, 49].

In the case of $N = 2$ and $d = 2$, $\beta(g)$ vanishes. This case corresponds to the classical XY model as mentioned above and there may be a Kosterlitz-Thouless transition. The Kosterlitz-Thouless transition point cannot be obtained by a perturbation expansion in g.

In two dimensions $d = 2$, $\beta(g)$ shows asymptotic freedom for $N > 2$. The coupling constant g approaches zero in high-energy limit $\mu \to \infty$ in a similar way to QCD. For $N = 1$, g increases as $\mu \to \infty$ as in the case of QED. When $d > 2$, there is a fixed point g_c:

$$g_c = \frac{\epsilon}{N-2}, \tag{118}$$

for $N > 2$. There is a phase transition for $N > 2$ and $d > 2$.

Let us consider the n-point function $\Gamma^{(n)}(k_i, g, \mu, H)$. The bare and renormalized n-point functions are introduced similarly and they are related by the renormalization constant Z

$$\Gamma_R^{(n)}(k_i, g, \mu, H) = Z^{n/2}\Gamma_B^{(n)}(k_i, g, \mu, H). \tag{119}$$

From the condition that the bare function $\Gamma_B^{(n)}$ is independent of μ, $\mu\, d\Gamma_B^{(n)}/d\mu = 0$, the renormalization group equation is followed

$$\left[\mu\frac{\partial}{\partial\mu} + \mu\frac{\partial g}{\partial\mu}\frac{\partial}{\partial g} - \frac{n}{2}\zeta(g) + \left(\frac{1}{2}\zeta(g) + \frac{1}{g}\beta(g) - (d-2)\right)H\frac{\partial}{\partial H}\right]\Gamma_R^{(n)}(k_i, g, \mu, H) = 0, \tag{120}$$

where we defined

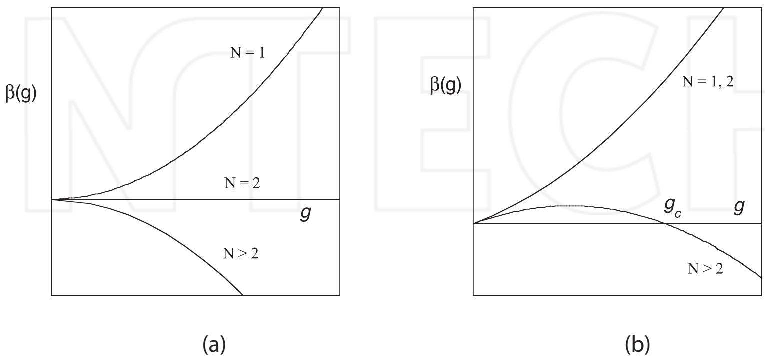

Figure 9. The beta function $\beta(g)$ as a function of g for $d = 2$ (a) and $d > 2$ (b). There is a fixed point for $N > 2$ and $d > 2$. $\beta(g)$ is negative for $d = 2$ and $N > 2$, which indicates that the model exhibits an asymptotic freedom.

$$\zeta(g) = \mu\frac{\partial}{\partial\mu}\ln Z = \beta(g)\frac{\partial}{\partial g}\ln Z. \tag{121}$$

From Eq. (113), $\zeta(g)$ is given by

$$\zeta(g) = (N-1)g + O(g^2). \tag{122}$$

Let us define the correlation length $\xi = \xi(g,\mu)$. Because the correlation length near the transition point will not depend on the energy scale, it should satisfy

$$\mu\frac{d}{d\mu}\xi(g,\mu) = \left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right)\xi(g,\mu) = 0. \tag{123}$$

We adopt the form $\xi = \mu^{-1}f(g)$ for a function $f(g)$, so that we have

$$\beta(g)\frac{df(g)}{dg} = f(g). \tag{124}$$

This indicates

$$f(g) = C\exp\left(\int_{g_*}^{g}\frac{1}{\beta(g')}dg'\right), \tag{125}$$

where C and g_* are constants. In two dimensions ($\epsilon = 0$), the beta function in Eq. (117) gives

$$\xi = C\mu^{-1}\exp\left(\frac{1}{N-2}\left(\frac{1}{g} - \frac{1}{g_*}\right)\right). \tag{126}$$

When $N > 2$, ξ diverges as $g \to 0$, namely, the mass proportional to ξ^{-1} vanishes in this limit. When $d > 2$ ($\epsilon > 0$), there is a finite-fixed point g_c. We approximate $\beta(g)$ near $g = g_c$ as

$$\beta(g) \approx a(g - g_c), \tag{127}$$

with $a < 0$, ξ is

$$\xi = \mu^{-1}\exp\left(\frac{1}{a}\ln\left\lfloor\frac{g-g_c}{g_*-g_c}\right\rfloor\right). \tag{128}$$

Near the critical point $g \approx g_c$, ξ is approximated as

$$\xi^{-1} \approx \mu\lfloor g - g_c\rfloor^{1/\lfloor a\rfloor}. \tag{129}$$

This means that $\xi \to \infty$ as $g \to g_c$. We define the exponent v by

http://dx.doi.org/10.5772/intechopen.68214

$$\xi^{-1} \approx \lfloor g - g_c \rfloor^{\nu}, \tag{130}$$

then we have

$$\nu = -\frac{1}{\beta'(g_c)}. \tag{131}$$

Since $\beta'(g_c) = \epsilon - 2(N-2)g_c = -\epsilon$, this gives

$$\frac{1}{\nu} = \epsilon + O(\epsilon^2) = d - 2 + O(\epsilon^2). \tag{132}$$

Including the higher-order terms, ν is given as

$$\frac{1}{\nu} = d - 2 + \frac{(d-2)^2}{N-2} + \frac{(d-2)^3}{2(N-2)} + O(\epsilon^4). \tag{133}$$

3.4. 2D quantum gravity

A similar renormalization group equation is derived for the two-dimensional quantum gravity. The space structure is written by the metric tensor $g_{\mu\nu}$ and the curvature R. The quantum gravity Lagrangian is

$$\mathcal{L} = -\frac{1}{16\pi G}\sqrt{g}R \tag{134}$$

where g is the determinant of the matrix $(g_{\mu\nu})$ and G is the coupling constant. The beta function for G was calculated as [78–81]

$$\beta(G) = \epsilon G - bG^2, \tag{135}$$

for $d = 2 + \epsilon$ with a constant b. This has the same structure as that for the non-linear sigma model.

4. Sine-Gordon model

4.1. Lagrangian

The two-dimensional sine-Gordon model has attracted a lot of attention [43–49, 82–91]. The Lagrangian of the sine-Gordon model is given by

$$\mathcal{L} = \frac{1}{2t_0}(\partial_\mu \phi)^2 + \frac{\alpha_0}{t_0}\cos\phi, \tag{136}$$

where ϕ is a real scalar field, and t_0 and α_0 are bare coupling constants. We also use the Euclidean notation in this section. The second term is the potential energy of the scalar field.

We adopt that t and α are positive. The renormalized coupling constants are denoted as t and α, respectively. The dimensions of t and α are $[t] = \mu^{2-d}$ and $[\alpha] = \mu^2$. The scalar field ϕ is dimensionless in this representation. The renormalization constants Z_t and Z_α are defined as follows

$$t_0 = t\mu^{2-d}Z_t,\ \alpha_0 = \ \alpha\mu^2 Z_\alpha. \tag{137}$$

Here, the energy scale μ is introduced so that t and α are dimensionless. The Lagrangian is written as

$$\mathcal{L} = \frac{\mu^{d-2}}{2tZ_t}(\partial_\mu\phi)^2 + \frac{\mu^d \alpha Z_\alpha}{tZ_t}\cos\phi. \tag{138}$$

We can introduce the renormalized field $\phi_B = \sqrt{Z_\phi}\phi_R$ where Z_ϕ is the renormalization constant. Then the Lagrangian is

$$\mathcal{L} = \frac{\mu^{d-2}Z_\phi}{2tZ_t}(\partial_\mu\phi)^2 + \frac{\mu^d \alpha Z_\alpha}{tZ_t}\cos\phi. \tag{139}$$

where ϕ denotes the renormalized field ϕ_R.

4.2. Renormalization of α

We investigate the renormalization group procedure for the sine-Gordon model on the basis of the dimensional regularization method. First consider the renormalization of the potential term. The lowest-order contributions are given by diagrams with tadpole contributions. We use the expansion $\cos\phi = 1 - \frac{1}{2}\phi^2 + \frac{1}{4!}\phi^4 - \cdots$. Then the corrections to the cosine term are evaluated as follows. The constant term is renormalized as

$$1 - \frac{1}{2}\langle\phi^2\rangle + \frac{1}{4!}\langle\phi^4\rangle - \ \cdots = 1 - \frac{1}{2}\langle\phi^2\rangle + \frac{1}{2}\left(\frac{1}{2}\langle\phi^2\rangle\right)^2 - \ \cdots = \exp\left(-\frac{1}{2}\langle\phi^2\rangle\right). \tag{140}$$

Similarly, the ϕ^2 is renormalized as

$$-\frac{1}{2}\phi^2 + \frac{1}{4!}6\langle\phi^2\rangle\phi^2 - \frac{1}{6!}15\cdot 3\langle\phi^2\rangle^2\phi^2 + \ \cdots = exp\left(-\frac{1}{2}\langle\phi^2\rangle\right)\left(-\frac{1}{2}\phi^2\right). \tag{141}$$

Hence the $\alpha Z_\alpha \cos(\sqrt{Z_\phi}\phi)$ is renormalized to

$$\alpha Z_\alpha \exp\left(-\frac{1}{2}Z_\phi\langle\phi^2\rangle\right)\cos\left(\sqrt{Z_\phi}\phi\right) \approx \alpha Z_\alpha\left(1 - \frac{1}{2}Z_\phi\langle\phi^2\rangle + \cdots\right)\cos\left(\sqrt{Z_\phi}\phi\right). \tag{142}$$

The expectation value $\langle\phi^2\rangle$ is regularized as

$$Z_\phi\langle\phi^2\rangle = t\mu^{2-d}\ Z_t\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m_0^2} = \ -\frac{t}{\epsilon}\frac{\Omega_d}{(2\pi)^d}, \tag{143}$$

where $d = 2+\epsilon$ and we included a mass m_0 to avoid the infrared divergence and Z_t=1 to this order. The constant Z_α is determined to cancel the divergence:

$$Z_\alpha = 1-\frac{t}{2}\frac{1}{\epsilon}\frac{\Omega_d}{(2\pi)^d}. \tag{144}$$

From the equations $\mu\,\partial t_0/\partial\mu = 0$ and $\mu\,\partial\alpha_0/\partial\mu = 0$, we obtain

$$\mu\frac{\partial t}{\partial\mu} = \ (d-2)t - t\mu\frac{\partial \ln Z_t}{\partial\mu}, \tag{145}$$

$$\mu\frac{\partial\alpha}{\partial\mu} = \ -2\alpha - \ \alpha\mu\frac{\partial \ln Z_\alpha}{\partial\mu} \tag{146}$$

The beta function for α reads

$$\beta(\alpha) \equiv \ \mu\frac{\partial\alpha}{\partial\mu} = \ -2\alpha + \ t\alpha\frac{1}{2}\frac{\Omega_d}{(2\pi)^d}, \tag{147}$$

where we set $\mu\,\partial\, t/\partial\mu = (d-2)t$ with $Z_t = 1$ up to the lowest order of α. The function $\beta(\alpha)$ has a zero at $t = t_c = 8\pi$.

4.3. Renormalization of the two-point function

Let us turn to the renormalization of the coupling constant t. The renormalization of t comes from the correction to p^2 term. The lowest-order two-point function is

$$\Gamma_B^{(2)(0)}(p) = \ \frac{1}{t_0}p^2 = \ \frac{1}{t\mu^{2-d}Z_t}p^2. \tag{148}$$

The diagrams that contribute to the two-point function are shown in **Figure 10** [88]. These diagrams are obtained by expanding the cosine function as $\cos\phi = 1-(1/2)\phi^{\,2}+\cdots$. First, we consider the Green's function,

$$G_0(x) = \ Z_\phi{<}\phi(x)\phi(0){>} = t\mu^{2-d}Z_t\int\frac{d^dp}{(2\pi)^d}p\frac{e^{ip\cdot x}}{p^2+m_0^2} = t\mu^{2-d}Z_t\frac{\Omega_d}{(2\pi)^d}K_0(m_0|x|), \tag{149}$$

where K_0 is the zeroth modified Bessel function and m_0 is introduced to avoid the infrared singularity. Because sinh $I-I = I^3/3!+\cdots$, the diagrams in **Figure 10** are summed up to give

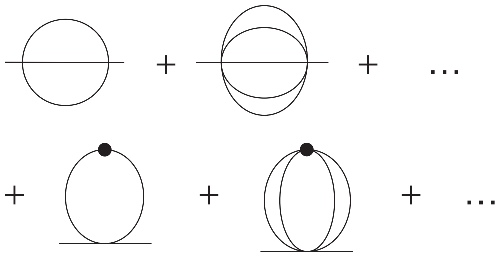

Figure 10. Diagrams that contribute to the two-point function.

$$\Sigma(p) = \int d^d x[e^{ip\cdot x}(\sinh I - I) - (\cosh I - 1)], \tag{150}$$

Where $I = G_0(x)$. Since $\sinh I - I \approx e^I/2$ and $\cosh I \approx e^I/2$, the diagrams in **Figure 10** lead to

$$\Gamma_B^{(2)c}(p) = -\frac{1}{2}\left(\frac{\alpha\mu^d Z_\alpha}{tZ_t}\right)^2 \int d^d x(e^{ip\cdot x} - 1)e^{G_0(x)}. \tag{151}$$

We use the expansion $e^{ip\cdot x} = 1 + ip\cdot x - (1/2)(p\cdot x)^2 + \cdots$, and keep the p_2 term. We denote the derivation of t from the fixed point $t_c = 8\pi$ as ν:

$$\frac{t}{8\pi} = 1 + \nu, \tag{152}$$

for $d = 2$. Using the asymptotic formula $K_0(x)^{\sim} - \gamma - \ln(x/2)$ for small x, we obtain

$$\begin{aligned}\Gamma_B^{(2)c}(p) &= \frac{1}{8}\left(\frac{\alpha\mu^d}{tZ_t}\right)^2 p^2(c_0 m_0^2)^{-2-2\nu}\Omega_d\int_0^\infty dx x^{d+1}\frac{1}{(x^2+a^2)^{2+2\nu}} \\ &= -\frac{1}{8}p^2\left(\frac{\alpha\mu^d}{tZ_t}\right)^2 (c_0 m_0^2)^{-2}\Omega_d\frac{1}{\epsilon} + \mathrm{O}(\nu) \\ &\approx -\frac{1}{t\mu^{2-d}Z_t}p^2\frac{1}{32}\alpha^2\mu^{d+2}(c_0 m_0^2)^{-2}\frac{1}{\epsilon} + O(\nu)\end{aligned} \tag{153}$$

where c_0 is a constant and $a = 1/\mu$ is a small cut-off. The divergence of α was absorbed by Z_α. Now the two-point function up to this order is

$$\Gamma_B^{(2)}(p) = \frac{1}{t\mu^{2-d}Z_t}\left[p^2 - \frac{1}{32}\alpha^2\mu^{d+2}(c_0 m_0^2)^{-2}\frac{1}{\epsilon}\right] \tag{154}$$

The renormalized two-point function is $\Gamma_R^{(2)} = Z_\phi\Gamma_B^{(2)}$. This indicates that

$$\frac{Z_\phi}{Z_t} = 1 + \frac{1}{32}\alpha^2\mu^{d+2}(c_0 m_0^2)^{-2}\frac{1}{\epsilon}. \tag{155}$$

Then, we can choose $Z_\phi = 1$ and

$$Z_t = 1 - \frac{1}{32}\alpha^2\mu^{d+2}(c_0 m_0^2)^{-2}\frac{1}{\epsilon}. \tag{156}$$

Z_t/Z_ϕ can be regarded as the renormalization constant of t up to the order of $\alpha 2$, and thus we do not need the renormalization constant Z_ϕ of the field ϕ. This means that we can adopt the bare coupling constant as $t_0 = t\mu^{2-d}\tilde{Z}_t$ with $\tilde{Z}_t = Z_t/Z_\phi$.

The renormalization function of t is obtained from the equation $\mu\,\partial\, t_0/\partial\mu = 0$ for $t_0 = t\mu^{2-d}Z_t$:

$$\begin{aligned} \beta(t) \;\equiv\; \mu\frac{\partial t}{\partial\mu} &= (d-2)t + \frac{1}{32}(c_0 m_0^2)^{-2}\frac{1}{\epsilon}\left(2\alpha\mu^{d+2}\mu\frac{\partial\alpha}{\partial\mu} + (d+2)\alpha^2\mu^{d+2}\right)t \\ &= (d-2)t + \frac{1}{32}\mu^{d+2}(c_0 m_0^2)^{-2}t\alpha^2 \end{aligned} \tag{157}$$

Because the finite part of $G_0(x \to 0)$ is given by $G_0(x \to 0) = -(1/2\pi)\ln(e^\gamma m_0/2\mu)$, we perform the finite renormalization of α as $\alpha \to \alpha c_0 m_0^2 a^2 = \alpha c_0 m_0^2 \mu^{-2}$. This results in

$$\beta(t) = (d-2)t + \frac{1}{32}t\alpha^2. \tag{158}$$

As a result, we obtain a set of renormalization group equations for the sine-Gordon model

$$\beta(\alpha) = \mu\frac{\partial\alpha}{\partial\mu} = -\alpha\left(2 - \frac{1}{4\pi}t\right), \tag{159}$$

$$\beta(t) = \mu\frac{\partial t}{\partial\mu} = (d-2)t + \frac{1}{32}t\alpha^2, \tag{160}$$

Since the equation for α is homogeneous in α, we can change the scale of α arbitrarily. Thus, the numerical coefficient of $t\alpha^2$ in $\beta(t)$ is not important.

4.4. Renormalization group flow

Let us investigate the renormalization group flow in two dimensions. This set of equations reduces to that of the Kosterlitz-Thouless (K-T) transition. We write $t = 8\pi(1+\nu)$, and set $x = 2\nu$ and $y = \alpha/4$. Then, the equations are

$$\mu \frac{\partial x}{\partial \mu} = y^2, \tag{161}$$

$$\mu \frac{\partial y}{\partial \mu} = xy, \tag{162}$$

These are the equations of K-T transition. We have

$$x^2 - y^2 = \text{const.} \tag{163}$$

The renormalization flow is shown in **Figure 11**. The Kosterlitz-Thouless transition is a beautiful transition that occurs in two dimensions. It was proposed that the transition was associated with the unbinding of vortices, that is, the K-T transition is a transition of the binding-unbinding transition of vortices.

The Kondo problem is also described by the same equations. In the s-d model, we put

$$x = \pi\beta J_z - 2, \; y = 2|J_\perp|\tau. \tag{164}$$

where J_z and $J_\perp(= J_x = J_y)$ are exchange coupling constants between the conduction electrons and the localized spin, and β is the inverse temperature. τ is a small cut-off with $\tau \propto 1/\mu$. The scaling equations for the s-d model are [53, 57]

$$\tau \frac{\partial x}{\partial \tau} = -\frac{1}{2} y^2, \tag{165}$$

$$\tau \frac{\partial y}{\partial \tau} = -\frac{1}{2} xy. \tag{166}$$

The Kondo effect occurs as a crossover from weakly correlated region to strongly correlated region. A crossover from weakly to strongly coupled systems is a universal and ubiquitous

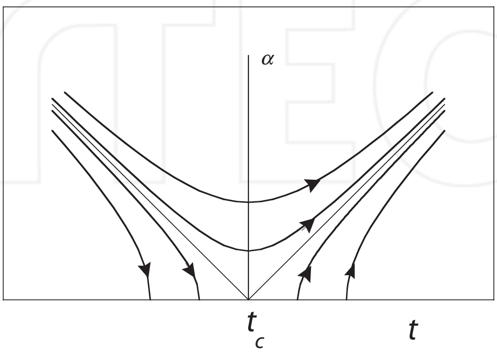

Figure 11. The renormalization group flow for the sine-Gordon model as $\mu \to \infty$.

phenomenon in the world. There appears a universal logarithmic anomaly as a result of the crossover.

5. Scalar quantum electrodynamics

We have examined the ϕ^4 theory and showed that there is a phase transition. This is a second-order transition. What will happen when a scalar field couples with the electromagnetic field? This issue concerns the theory of a complex scalar field ϕ interacting with the electromagnetic field A_μ, called the scalar quantum electrodynamics (QED). The Lagrangian is

$$\mathcal{L} = \frac{1}{2}|(D_\mu\phi)|^2 - \frac{1}{4}g(|\phi|^2)^2 - \frac{1}{4}F_{\mu\nu}^2, \tag{167}$$

where g is the coupling constant and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. D_μ is the covariant derivative given as

$$D_\mu = \partial_\mu - ieA_\mu, \tag{168}$$

with the charge e. The scalar field ϕ is an N component complex scalar field such as $\phi = (\phi_1, \cdots, \phi_N)$. This model is actually a model of a superconductor. The renormalization group analysis shows that this model exhibits a first-order transition near four dimensions $d = 4 - \epsilon$ when $2N < 365$ [92–96]. Coleman and Weinberg first considered the scalar QED model in the case $N = 1$. They called this transition the dimensional transmutation. The result based on the ϵ-expansion predicts that a superconducting transition in a magnetic field is a first-order transition. This transition may be related to a first-order transition in a high magnetic field [97].

The bare and renormalized fields and coupling constants are defined as

$$\phi_0 = \sqrt{Z_\phi}\phi, \tag{169}$$

$$g_0 = \frac{Z_4}{Z_\phi^2}g\mu^{4-d}, \tag{170}$$

$$e_0 = \frac{Z_e}{\sqrt{Z_A Z_\phi}}e, \tag{171}$$

$$A_{\mu 0} = \sqrt{Z_A}A_\mu, \tag{172}$$

where ϕ, g, e and A_μ are renormalized quantities. We have four renormalization constants. Thanks to the Ward identity

$$Z_e = Z_A, \tag{173}$$

three renormalization constants should be determined. We show the results:

$$Z_\phi = 1 + \frac{3}{8\pi^2\epsilon}e^2, \tag{174}$$

$$Z_A = 1 - \frac{2N}{48\pi^2\epsilon}e^2, \tag{175}$$

$$Z_g = 1 + \frac{2N+8}{8\pi^2\epsilon}g + \frac{3}{8\pi^2\epsilon}\frac{1}{g}e^4. \tag{176}$$

The renormalization group equations are given by

$$\mu\frac{\partial e^2}{\partial\mu} = -\epsilon e^2 + \frac{N}{24\pi^2}e^4, \tag{177}$$

$$\mu\frac{\partial g}{\partial\mu} = -\epsilon g + \frac{N+4}{4\pi^2}g^2 + \frac{3}{8\pi^2}e^4 - \frac{3}{4\pi^2}e^2 g. \tag{178}$$

The fixed point is given by

$$e_c = \frac{24}{N}\pi^2\epsilon, \tag{179}$$

$$g_c = \epsilon\frac{2\pi^2}{N+4}\left\{1 + \frac{18}{N} \pm \frac{\left(n^2 - 360n - 2160\right)^{1/2}}{n}\right\}, \tag{180}$$

where $n = 2N$. The square root $\delta \equiv \left(n^2 - 360n - 2160\right)^{1/2}$ is real when $2N > 365$. This indicates that the zero of a set of beta functions exists when N is sufficiently large as long as $2N > 365$. Hence there is no continuous transition when N is small, $2N \leq 365$, and the phase transition is first-order.

There are also calculations up to two-loop-order for scalar QED [98, 99]. This model is also closely related with the phase transition from a smectic-A to a nematic liquid crystal for which a second-order transition was reported [100]. When N is large as far as $2N > 365$, the transition becomes second-order. Does the renormalization group result for the scalar QED contradict with second-order transition in superconductors? This subject has not been solved yet. A possibility of second-order transition was investigated in three dimensions by using the renormalization group theory [101]. An extra parameter c was introduced in [101] to impose a relation between the external momentum p and the momentum q of the gauge field as $q = p/c$. It was shown that when $c > 5.7$, we have a second-order transition. We do not think that it is clear whether the introduction of c is justified or not.

6. Summary

We presented the renormalization group procedure for several important models in field theory on the basis of the dimensional regularization method. The dimensional method is very useful and the divergence is separated from an integral without ambiguity. We invested three fundamental models in field theory: ϕ4 theory, non-linear sigma model and sine-Gordon model. These models are often regarded as an effective model in understanding physical phenomena. The renormalization group equations were derived in a standard way by regularizing the ultraviolet divergence. The renormalization group theory is useful in the study of various quantum systems.

The renormalization means that the divergences, appearing in the evaluation of physical quantities, are removed by introducing the finite number of renormalization constants. If we need infinite number of constants to cancel the divergences for some model, that model is called unrenormalizable. There are many renormalizeable field theoretic models. We considered three typical models among them. The idea of renormalization group theory arises naturally from renormalization. The dependence of physical quantities on the renormalization energy scale easily leads us to the idea of renormalization group.

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