PTEP

Prog. Theor. Exp. Phys. **2021**, 033A01 (19 pages)
DOI: 10.1093/ptep/ptab026

Renormalization group theory of the generalized multi-vertex sine-Gordon model

Takashi Yanagisawa*

Electronics and Photonics Research Institute, National Institute of Advanced Industrial Science and Technology, 1-1-1 Umezono, Tsukuba, Ibaraki 305-8568, Japan
*E-mail: t-yanagisawa@aist.go.jp

Received January 18, 2021; Accepted February 17, 2021; Published February 25, 2021

We investigate the renormalization group theory of the generalized multi-vertex sine-Gordon model by employing the dimensional regularization method and also the Wilson renormalization group method. The vertex interaction is given by $\cos(k_j \cdot \phi)$, where k_j ($j = 1, 2, \ldots, M$) are momentum vectors and ϕ is an N-component scalar field. The beta functions are calculated for the sine-Gordon model with multiple cosine interactions. The second-order correction in the renormalization procedure is given by the two-point scattering amplitude for tachyon scattering. We show that new vertex interaction with the momentum vector k_ℓ is generated from two vertex interactions with vectors k_i and k_j when k_i and k_j meet the condition $k_\ell = k_i \pm k_j$, called the triangle condition. A further condition $k_i \cdot k_j = \pm 1/2$ is required within the dimensional regularization method. The renormalization group equations form a set of closed equations when $\{k_j\}$ form an equilateral triangle for $N = 2$ or a regular tetrahedron for $N = 3$. The Wilsonian renormalization group method gives qualitatively the same result for beta functions.

Subject Index A40, B32, B34, I63

1. Introduction

The sine-Gordon model is an interesting universal model that appears as an effective model in various fields of physics [1–17]. The two-dimensional (2D) sine-Gordon model can be mapped to the Coulomb gas model that has logarithmic Coulomb interaction [18,19]. The 2D sine-Gordon model has been investigated using several methods, in particular the renormalization group method. The physics of the sine-Gordon model is closely related to that of the Kosterlitz–Thouless transition of the 2D XY model [20,21].

The sine-Gordon model is a model of a scalar field under periodic potential, and can be generalized in several ways. The massive chiral model is regarded as a generalization of the sine-Gordon model where the potential term $\mathrm{Tr}(g+g^{-1})$ is considered for g in a gauge group (Lie group) G ($g \in G$) [22–24]. The chiral model was generalized to include the Wess–Zumino term as the Wess–Zumino–Witten model [25–28]. The other way of generalizing is to include the potential terms of high-frequency modes [29,30]. A generalized potential term is given as

$$V = \frac{1}{t}\sum_{n=1}^{L}\alpha_n \cos(n\phi), \tag{1}$$

where ϕ is a one-component scalar field and L is an integer. In the Wilson renormalization group method, the cosine potential $\cos((n-m)\phi)$ is generated from $\cos(n\phi)$ and $\cos(m\phi)$ in the second-order perturbation. Thus there will be a correction to the beta function of α_n in the form $\alpha_\ell\alpha_m$, with

$n = |\ell - m|$. For the hyperbolic sine-Gordon model, α_n has a correction from α_ℓ and α_m satisfying $n = \ell + m$ [31].

This kind of model can be generalized to a multi-component scalar field. In this paper we investigate a multi-vertex sine-Gordon model with multiple cosine potentials. The cosine vertex interaction is given by $\cos\left(\sum_\ell k_{j\ell}\phi_\ell\right)$, where $\phi = (\phi_1, \ldots, \phi_N)$ is a scalar field and $k_j = (k_{j1}, \ldots, k_{jN})$ ($j = 1, \ldots, M$) are momentum vectors of real numbers; k_j represents the direction of oscillation of the field ϕ. The model for $M = 3$ was considered in Ref. [32]. The condition to generate a new vertex interaction shown above is generalized to $k_n = k_\ell \pm k_m$. This is called the triangle condition in this paper. We further consider a generalized multi-vertex sine-Gordon model.

It has been pointed out that there is a close relation between the sine-Gordon model and string propagation in a tachyon background [33]. In fact, two-vertex correction in the renormalization procedure is given by the two-point scattering amplitude for tachyon scattering in second-order perturbation theory. The multi-vertex correction will be given by the multi-point tachyon scattering amplitude.

This paper is organized as follows. In Sect. 3 we present the multi-vertex sine-Gordon model. We show the renormalization procedure based on the dimensional regularization method in Sect. 4. We apply the Wilsonian renormalization group method to our model in Sect. 5. We consider the generalized multi-vertex sine-Gordon model and calculate the beta functions in Sect. 6, and provide a summary in the last section.

2. Multi-vertex sine-Gordon model

We consider an N-component real scalar field $\phi = (\phi_1, \ldots, \phi_N)$. The model is a d-dimensional generalized multi-vertex sine-Gordon model given by

$$\mathcal{L} = \frac{1}{2t_0}\left(\partial_\mu \phi\right)^2 + \frac{1}{t_0}\sum_{j=1}^{M} \alpha_{0j} \cos\left(k_j \cdot \phi\right), \tag{2}$$

where $t_0 (> 0)$ and α_{0j} ($j = 1, \ldots, M$) are bare coupling constants, and k_j ($j = 1, \ldots, M$) are N-component constant vectors. We use the notation $(\partial_\mu \phi)^2 = \sum_j (\partial_\mu \phi_j)^2$ and $k_j \cdot \phi = \sum_\ell k_{j\ell}\phi_\ell$ for $k_j = (k_{j1}, \ldots, k_{jN})$. We use the Euclidean notation in this paper. The second term is the potential energy with multiple cosine interactions. The dimensions of t_0 and α_{0j} are given as $[t_0] = \mu^{2-d}$ and $[\alpha_{0j}] = \mu^2$ for the energy scale parameter μ. The analysis is performed near two dimensions, $d = 2$. We introduce the renormalized coupling constants t and α_j, where the renormalization constants are defined as

$$t_0 = t\mu^{2-d} Z_t, \qquad \alpha_{0j} = \alpha_j \mu^2 Z_{\alpha_j}, \tag{3}$$

where we set that t and α_j are dimensionless constants. The renormalized field ϕ_R is introduced with the renormalization constant Z_ϕ as

$$\phi = \sqrt{Z_\phi}\phi_R. \tag{4}$$

In the following, ϕ denotes the renormalized field ϕ_R for simplicity. Then, the Lagrangian density is given as

$$\mathcal{L} = \frac{\mu^{d-2}Z_\phi}{2tZ_t}\left(\partial_\mu\phi\right)^2 + \frac{\mu^d}{tZ_t}\sum_j Z_{\alpha_j}\alpha_j\cos(k_j\cdot\phi). \tag{5}$$

We examine the renormalization group procedure for this model in Sects. 3 and 4. We also investigate the component dependence of renormalization in Sect. 5 by generalizing the model as follows:

$$\mathcal{L} = \sum_\ell \frac{\mu^{d-2}Z_\phi}{2t_\ell Z_{t_\ell}}(\partial_\mu\phi_\ell)^2 + \sum_j \frac{\mu^d\alpha_j Z_{\alpha_j}}{t_j Z_{t_j}}\cos\left(\sqrt{Z_\phi}k_j\cdot\phi\right). \tag{6}$$

We need some conditions so that we have one fixed point for t. For this purpose we normalize k vectors as

$$k_j^2 = \sum_{\ell=1}^N k_{j\ell}^2 = 1, \qquad (j = 1,\ldots,M). \tag{7}$$

From two vertices with momentum vectors k_i and k_j, a new vertex with momentum k_m is generated when the triangle condition is satisfied:

$$k_m = k_i \pm k_j. \tag{8}$$

We assume that the set $\{\alpha_j\}$ includes all vertices that will be generated from multi-vertex interactions with each other. For a triangle or a regular polyhedron which is composed of equilateral triangles, M becomes finite since $\{k_j\}$ form a finite set. For example, we will consider an equilateral triangle or a regular tetrahedron. For an equilateral triangle ($N = 2$, $M = 3$) or a regular tetrahedron ($N = 3$, $M = 6$), we further have

$$\sum_{j=1}^M k_{j\ell}^2 = C(M) \qquad \text{for } \ell = 1,\ldots,N, \tag{9}$$

where $C(M)$ is a constant depending on M. For an equilateral triangle for a three-component scalar field ($N = 3$ and $M = 3$), this relation does not hold. These conditions will be explained in the following sections.

3. Renormalization by dimensional regularization

We evaluate the beta functions for the multi-vertex sine-Gordon model by using the dimensional regularization method [34–36].

3.1. *Tadpole renormalization of α_j*

The lowest-order contributions to the renormalization of α_j are given by tadpole diagrams. Using the expansion $\cos\phi = 1 - \frac{1}{2}\phi^2 + \frac{1}{4!}\phi^4 - \cdots$, the cosine potential is renormalized as

$$\cos\left(\sqrt{Z_\phi}k_j\cdot\phi\right) \;\to\; \left(1 - \frac{1}{2}Z_\phi\langle(k_j\cdot\phi)^2\rangle + \cdots\right)\cos\left(\sqrt{Z_\phi}k_j\cdot\phi\right). \tag{10}$$

$\langle\phi^2\rangle$ is regularized as

$$\langle\phi_\ell^2\rangle = \frac{t\mu^{2-d}Z_t}{Z_\phi}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m_0^2} = -\frac{t\mu^{2-d}Z_t}{Z_\phi}\frac{1}{\epsilon}\frac{\Omega_d}{(2\pi)^d} \tag{11}$$

for $d = 2+\epsilon$, where a mass m_0 is introduced to avoid the infrared divergence. We set $Z_t = 1$ in the lowest order of t. We adopt that $\langle\phi_i\phi_\ell\rangle = \delta_{i\ell}\langle\phi_i^2\rangle$ and $\langle\phi_\ell^2\rangle$ is independent of ℓ. Then the renormalization of the potential term is given as

$$\begin{aligned}&\alpha_j Z_{\alpha_j}\left(1-\frac{1}{2}Z_\phi k_j^2\langle\phi_1^2\rangle+\cdots\right)\cos\left(\sqrt{Z_\phi}k_j\cdot\phi\right)\\ &= \alpha_j Z_{\alpha_j}\left(1+\frac{1}{2}k_j^2\frac{1}{\epsilon}t\mu^{2-d}\frac{\Omega_d}{(2\pi)^d}+\cdots\right)\cos\left(\sqrt{Z_\phi}k_j\cdot\phi\right).\end{aligned} \tag{12}$$

The renormalization constant Z_{α_j} is determined as

$$Z_{\alpha_j} = 1-\frac{t}{4\pi\epsilon}k_j^2. \tag{13}$$

Since the bare coupling constant $\alpha_{0j} = \alpha_j\mu^2 Z_{\alpha_j}$ is independent of μ, we have

$$\beta(\alpha_j) := \mu\frac{\partial\alpha_j}{\partial\mu} = -2\alpha_j+\frac{1}{4\pi}k_j^2 t\alpha_j. \tag{14}$$

The beta function of α_j has a zero at

$$t = t_{cj} = 8\pi/k_j^2 = 8\pi, \tag{15}$$

since $k_j^2 = 1$.

3.2. *Vertex–vertex interaction*

We investigate the corrections to t and α_j from vertex–vertex interactions. The second-order contribution $I^{(2)}$ to the action is given by

$$\begin{aligned}I^{(2)} &= -\frac{1}{2}\left(\frac{\mu^d}{tZ_t}\right)^2\int d^dxd^dx'\sum_{ij}\alpha_i\alpha_j Z_{\alpha_i}Z_{\alpha_j}\cos\left(\sqrt{Z_\phi}k_i\cdot\phi(x)\right)\cos\left(\sqrt{Z_\phi}k_j\cdot\phi(x')\right)\\ &= -\frac{1}{4}\left(\frac{\mu^d}{tZ_t}\right)^2\int d^dxd^dx'\sum_{ij}\alpha_i\alpha_j Z_{\alpha_i}Z_{\alpha_j}\Big[\cos\left(\sqrt{Z_\phi}(k_i\cdot\phi(x)-k_j\cdot\phi(x'))\right)\\ &\quad+\cos\left(\sqrt{Z_\phi}(k_i\cdot\phi(x)+k_j\cdot\phi(x'))\right)\Big].\end{aligned} \tag{16}$$

We first examine the first term, denoted as $I_1^{(2)}$:

$$I_1^{(2)} = -\frac{1}{4}\left(\frac{\mu^d}{tZ_t}\right)^2\int d^dxd^dx'\sum_{ij}\alpha_i\alpha_j Z_{\alpha_i}Z_{\alpha_j}\cos\left(\sqrt{Z_\phi}(k_i\cdot\phi(x)-k_j\cdot\phi(x'))\right). \tag{17}$$

We evaluate the renormalization of the cosine term by calculating $\langle(k_i\cdot\phi-k_j\cdot\phi)^2\rangle$. We adopt that $\langle\phi_\ell(x)\phi_m(x')\rangle = \delta_{\ell m}\langle\phi_\ell(x)\phi_\ell(x')\rangle$, and $\langle\phi_\ell(x)\phi_\ell(x')\rangle$ is independent of ℓ: $\langle\phi_\ell(x)\phi_\ell(x')\rangle = \langle\phi_1(x)\phi_1(x')\rangle$. Then

$$\langle(k_i\cdot\phi-k_j\cdot\phi)^2\rangle = \sum_\ell\left[k_{i\ell}^2\langle\phi_\ell(x)^2\rangle+k_{j\ell}^2\langle\phi_\ell(x')^2\rangle-2k_{i\ell}k_{j\ell}\langle\phi_\ell(x)\phi_\ell(x')\rangle\right]. \tag{18}$$

$I_1^{(2)}$ is renormalized as

$$I_1^{(2)} = -\frac{1}{4}\left(\frac{\mu^d}{tZ_t}\right)^2 \int d^dxd^dx' \Bigg[\sum_i \alpha_i^2 Z_{\alpha_i}^2 \exp\left(-Z_\phi k_i^2 \langle \phi_1(x)^2\rangle + Z_\phi k_i^2 \langle \phi_1(x)\phi_1(x')\rangle\right) \times \cos\left(\sqrt{Z_\phi}(k_i\cdot(\phi(x)-\phi(x')))\right) + \sum_{i\neq j}\alpha_i\alpha_j Z_{\alpha_i}Z_{\alpha_j} \exp\Big[-\frac{Z_\phi}{2}(k_i^2\langle\phi_1(x)^2\rangle + k_j^2\langle\phi_1(x')^2\rangle) + Z_\phi k_i\cdot k_j\langle\phi_1(x)\phi_1(x')\rangle\Big] \times \cos\left(\sqrt{Z_\phi}(k_i\cdot\phi(x) - k_j\cdot\phi(x'))\right)\Bigg]. \tag{19}$$

The two-point function is written as

$$\langle\phi_\ell(x)\phi_\ell(y)\rangle = \frac{t\mu^{2-d}Z_t}{Z_\phi}\int\frac{d^dp}{(2\pi)^d}\frac{e^{ip\cdot(x-y)}}{p^2+m_0^2} = \frac{t\mu^{2-d}Z_t}{Z_\phi}\frac{\Omega_d}{(2\pi)^d}K_0(m_0|x-y|), \tag{20}$$

where we introduced m_0 to avoid the infrared divergence and K_0 is the zeroth modified Bessel function.

3.3. *Renormalization of t*

The first term of $I_1^{(2)}$ gives a contribution to the renormalization of the coupling constant t. Since $K_0(m_0r)$ increases as $r\to 0$, we can expand in terms of r. By using $\cos\left(\sqrt{Z_\phi}k_i\cdot(\phi(x)-\phi(x+\mathbf{r}))\right)\simeq 1-(1/2)Z_\phi(r_\mu\partial_\mu(k_i\cdot\phi(x)))^2$, where $\partial_\mu=\partial/\partial x_\mu$, the first term $I_{1a}^{(2)}$ of $I_1^{(2)}$ is written as

$$I_{1a}^{(2)} \simeq \frac{1}{4}\left(\frac{\mu^d}{tZ_t}\right)^2\int d^dxd^dr\sum_i\alpha_i^2Z_{\alpha_i}^2\frac{1}{4}Z_\phi(\partial_\mu\tilde{\phi}_i)^2r^2\exp\left(-Z_\phi k_i^2\langle\phi_1^2\rangle + tZ_tk_i^2\frac{\Omega_d}{(2\pi)^d}K_0(m_0r)\right), \tag{21}$$

where $\tilde{\phi}_i=\sum_\ell k_{i\ell}\phi_\ell$. If $(k_{j\ell})\in SO(N)$ (with $M=N$), we have $\sum_i(\partial_\mu\phi_i)^2=\sum_i(\partial_\mu\tilde{\phi}_i)^2$. In general, we have

$$\sum_i(\partial_\mu\tilde{\phi}_i)^2 = \sum_{i\ell}k_{i\ell}^2(\partial_\mu\phi_\ell)^2 + \sum_{i,\ell\neq m}k_{i\ell}k_{im}\partial_\mu\phi_\ell\partial_\mu\phi_m. \tag{22}$$

As mentioned in Sect. 2, we consider the case where $\{k_j\}$ form an equilateral triangle ($M=3$) or a regular tetrahedron ($M=6$), and we obtain $\sum_i k_{i\ell}^2=$ constant $\equiv C$ depending on M, such as $C=3/2$ for $M=3$ and $N=2$, and $C=2$ for $M=6$ and $N=3$. In this case,

$$\sum_i(\partial_\mu\tilde{\phi}_i)^2 = C\sum_i(\partial_\mu\phi_i)^2 + \sum_{i,\ell\neq m}k_{i\ell}k_{im}\partial_\mu\phi_\ell\partial_\mu\phi_m. \tag{23}$$

In order to recover the kinetic term in the original action, we use the approximation

$$\sum_i\alpha_i^2Z_{\alpha_i}^2(\partial_\mu\tilde{\phi}_i)^2 \to \frac{1}{M}\sum_i\alpha_i^2\cdot C(\partial_\mu\phi)^2 \equiv \langle\alpha_i^2\rangle C(\partial_\mu\phi)^2; \tag{24}$$

otherwise, the renormalization of the kinetic term becomes complicated since we must introduce $\{t_i\}$ that depend on components of ϕ. This may not be essential for the renormalization group flow. We discuss this point later.

$\langle\phi_1^2\rangle$ is evaluated as

$$\langle\phi_1^2\rangle = \frac{t\mu^{2-d}Z_t}{Z_\phi}\frac{\Omega_d}{(2\pi)^d}K_0(m_0 a), \tag{25}$$

where a is a small cutoff. The r-integration is calculated as

$$\begin{aligned} J_j &:= \int d^d r r^2 \exp\left(tk_j^2 \frac{\Omega_d}{(2\pi)^d}K_0(m_0\sqrt{r^2+a^2})\right) \\ &\simeq \Omega_d \int dr r^{d+1}\left(\frac{1}{cm_0^2(r^2+a^2)}\right)^{\frac{t}{4\pi}k_j^2}, \end{aligned} \tag{26}$$

where $c = (e^\gamma/2)^2$. We put

$$d = 2+\epsilon, \tag{27}$$

and

$$\frac{t}{8\pi} = 1+v, \tag{28}$$

since we normalize $k_j^2 = 1$. Then we have

$$\begin{aligned} J_j &= \Omega_d (cm_0^2)^{-2}\int_0^\infty dr r^{d+1}\frac{1}{(r^2+a^2)^{2+2v}} \\ &= -\Omega_d (cm_0^2)^{-2}\frac{1}{\epsilon} + O(v). \end{aligned} \tag{29}$$

This indicates that

$$\begin{aligned} I_{1a}^{(2)} &= -\frac{C}{8}\left(\frac{\mu^d}{tZ_t}\right)^2 \langle\alpha_i^2\rangle \exp\left(-Z_\phi\langle\phi_1^2\rangle\right)\Omega_d(cm_0^2)^{-2}\frac{1}{\epsilon}\int d^d x \frac{1}{2}Z_\phi k_i^2(\partial_\mu\phi)^2 + O(v_i) \\ &= -\frac{C}{8}\left(\frac{\mu^d}{tZ_t}\right)^2 \langle\alpha_i^2\rangle (cm_0^2 a^2)^{tZ_t/4\pi}\Omega_d(cm_0^2)^{-2}\frac{1}{\epsilon}\int d^d x \frac{1}{2}Z_\phi(\partial_\mu\phi)^2 + O(v) \\ &= -\frac{C}{8}\langle\alpha_i^2\rangle\frac{\Omega_d}{8\pi}\mu^{d+2}a^4\frac{1}{\epsilon}\int d^d x \frac{\mu^{d-2}Z_\phi}{2tZ_t}(\partial_\mu\phi)^2 + O(v). \end{aligned} \tag{30}$$

Then we choose

$$Z_t = 1 - \frac{C}{32}\langle\alpha_i^2\rangle\mu^{d+2}a^4\frac{1}{\epsilon}, \tag{31}$$

where we set $Z_{\alpha_i} = 1$ to the lowest order of α_i.

Since the bare coupling constant $t_0 = t\mu^{2-d}Z_t$ is independent of the energy scale μ, we have $\mu\partial t_0/\partial\mu = 0$. This results in

$$\beta(t) := \mu\frac{\partial t}{\partial\mu} = (d-2)t - t\mu\frac{\partial \ln Z_t}{\partial\mu}$$
$$= (d-2)t + \frac{C}{32}t\langle\alpha_i^2\rangle, \tag{32}$$

where we used $\mu\partial\alpha_i/\partial\mu = -2(\alpha_i - t/8\pi)$, neglecting terms of the order of $t^2\alpha_i^2$, and we put $a = \mu^{-1}$.

3.4. Vertex–vertex correction to α_j

We consider the second term in $I_{1b}^{(2)}$ that contains multi-vertex interaction:

$$I_{1b}^{(2)} := -\frac{1}{4}\left(\frac{\mu^d}{tZ_t}\right)^2 \int d^dxd^dx' \sum_{i\neq j}\alpha_i\alpha_j Z_{\alpha_i}Z_{\alpha_j}\exp\Big[-\frac{Z_\phi}{2}\left(k_i^2\langle\phi_1(x)^2\rangle + k_j^2\langle\phi_1(x')^2\rangle\right)$$
$$+ Z_\phi k_i\cdot k_j\langle\phi_1(x)\phi_1(x')\rangle\Big]\cos\left(\sqrt{Z_\phi}(k_i\cdot\phi(x) - k_j\cdot\phi(x'))\right). \tag{33}$$

Let us examine the integral given by

$$J_{ij} := \int d^dr\exp\left(Z_\phi k_i\cdot k_j\langle\phi(x)_1\phi_1(x+r)\rangle\right)$$
$$= \int d^dr\exp\left(k_i\cdot k_j t\mu^{2-d}Z_t\frac{\Omega_d}{(2\pi)^d}K_0(m_0r)\right)$$
$$\simeq \Omega_d\int_0^\infty drr^{d-1}\left(\frac{1}{cm_0^2(r^2+a^2)}\right)^{tk_i\cdot k_j/4\pi}$$
$$= \Omega_d\left(\frac{1}{cm_0^2a^2}\right)^{tk_i\cdot k_j}\frac{a^{d/2}}{2}\frac{1}{\Gamma(tk_i\cdot k_j/4\pi)}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{t}{4\pi}k_i\cdot k_j - \frac{d}{2}\right), \tag{34}$$

where the cutoff a is introduced. We put $t = 8\pi(1+v)$, then we have a divergence near two dimensions when

$$k_i\cdot k_j = 1/2. \tag{35}$$

This means that the two vectors k_i and k_j form an equilateral triangle. When k_i and k_j satisfy this condition, we have

$$J_{ij} = -\Omega_d(cm_0^2)^{-1}\frac{1}{\epsilon} + O(v). \tag{36}$$

Then we obtain

$$I_{1b}^{(2)} \simeq \frac{1}{4}\left(\frac{\mu^d}{tZ_t}\right)^2\sum_{i\neq j}\alpha_i Z_{\alpha_i}\alpha_j Z_{\alpha j}\left(cm_0^2a^2\right)^{t/4\pi}\frac{1}{\epsilon}\Omega_d(cm_0^2)^{-1}\int d^dx\cos\left(\sqrt{Z_\phi}(k_i-k_j)\cdot\phi(x)\right)$$
$$\simeq \frac{1}{\epsilon}\frac{cm_0^2}{16}\sum_{i\neq j}\alpha_i\alpha_j\frac{1}{tZ_t}\mu^{2d}a^4\int d^dx\cos\left(\sqrt{Z_\phi}(k_i-k_j)\cdot\phi(x)\right). \tag{37}$$

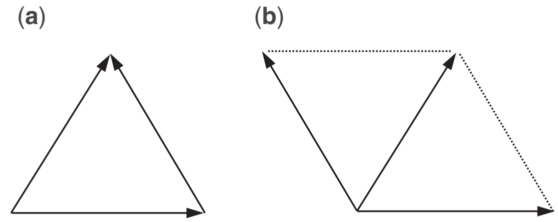

Fig. 1. Triangles formed by wave vectors k_i, k_j, and k_ℓ for (a) $k_i \cdot k_j = 1/2$ and (b) $k_i \cdot k_j = -1/2$.

Let k_ℓ be a vector such that k_i, k_j, and k_ℓ form an equilateral triangle:

$$k_i - k_j = k_\ell. \tag{38}$$

Then the potential term with coefficient α_ℓ has the correction as

$$\frac{\mu^d}{tZ_t}\alpha_\ell Z_{\alpha_\ell}\left(1 + \frac{t}{4\pi\epsilon} + \frac{1}{16\epsilon}\frac{\alpha_i\alpha_j}{\alpha_\ell}\mu^d cm_0^2 a^4\right)\cos\left(\sqrt{Z_\phi}k_\ell \cdot \phi\right). \tag{39}$$

We choose the renormalization constant Z_{α_ℓ} as

$$Z_{\alpha_\ell} = 1 - \frac{t}{4\pi\epsilon} - \frac{1}{16\epsilon}\frac{\alpha_i\alpha_j}{\alpha_\ell}\mu^d cm_0^2 a^4. \tag{40}$$

This leads to the beta function $\beta(\alpha_\ell)$ with correction as

$$\beta(\alpha_\ell) = -2\alpha_\ell\left(1 - \frac{t}{8\pi}\right) + \frac{1}{8}cm_0^2 a^2 \alpha_i\alpha_j. \tag{41}$$

Since the coefficient of the correction term is dependent on the cutoff parameters, we choose $cm_0^2 a^2 = 1$ to have

$$\beta(\alpha_\ell) = -2\alpha_\ell\left(1 - \frac{t}{8\pi}\right) + \frac{1}{16}\alpha_i\alpha_j. \tag{42}$$

There is also a contribution from the second term in $I^{(2)}$ where k_j is replaced by $-k_j$. In this case, vertices with $k_i \cdot k_j = -1/2$ generate a new vertex with k_ℓ satisfying

$$k_i + k_j = k_\ell. \tag{43}$$

In the dimensional regularization method, two vertices satisfying $k_i \cdot k_j = \pm 1/2$ generate a new vertex with $k_i \mp k_j$ (see Fig. 1). As a result, the beta function for α_ℓ reads

$$\beta(\alpha_\ell) = -2\alpha_\ell\left(1 - \frac{t}{8\pi}\right) + \frac{1}{16}\sum_{ij}{}' \alpha_i\alpha_j, \tag{44}$$

where the summation should take for those satisfying $k_\ell = k_i \pm k_j$ $(i, j, \ell = 1, \ldots, N)$.

When k_1, k_2, and k_3 form an equilateral triangle ($M = 3$), the renormalization group equations for α_1, α_2, and α_3 are closed within three equations. When k_1, k_2, …, k_6 form a regular tetrahedron ($M = 6$), we again have a closed set of equations for α_j ($j = 1, 2, \ldots, 6$). In the Wilsonian method, the same beta equation for α_ℓ is obtained, as discussed in the next section. In the Wilson renormalization method, however, a new vertex $k_i \mp k_j$ is generated from any two vectors k_i and k_j except in the case $k_i \cdot k_j = 0$.

3.5. *Relation to the tachyon scattering amplitude in a bosonic string theory*

The two-vertex correction J_{ij} is related to the tachyon scattering amplitude in a bosonic string theory. The n-point scattering amplitude for tachyon scattering is given as [37,38]

$$\begin{aligned} A_n &:= \int d\mu \int DX \exp\left[-\frac{1}{4\pi\alpha'} \int (\partial_z X_\mu \partial_{\bar{z}} X^\mu) d^2 z + i \sum_{i=1}^{n} k_{i\mu} X^\mu \right] \\ &= \int d\mu \prod_{1\le i<j\le n} |z_i - z_j|^{2\alpha' k_i \cdot k_j}, \end{aligned} \tag{45}$$

where the integration with the measure $d\mu$ is an integral over the various z_i. If we assume the correspondence

$$2\pi\alpha' = t, \tag{46}$$

the z_i dependence of the amplitude A_2 agrees with J_{ij} where $J_{ij} \sim \int d^d r A_{ij}(r)^{-1}$ with $A_{ij}(r) = r^{\alpha' k_i \cdot k_j} = r^{t k_i \cdot k_j / 2\pi}$. The vertex–vertex renormalization is given by the amplitude for tachyon scattering.

3.6. *Renormalization group flow*

For an equilateral triangle configuration of $\{k_i\}$ with $M = 3$ and $N = 2$, the equations read

$$\begin{aligned} \mu \frac{\partial \alpha_1}{\partial \mu} &= -2\alpha_1 \left(1 - \frac{t}{8\pi}\right) + \frac{1}{16}\alpha_2\alpha_3, \\ \mu \frac{\partial \alpha_2}{\partial \mu} &= -2\alpha_2 \left(1 - \frac{t}{8\pi}\right) + \frac{1}{16}\alpha_3\alpha_1, \\ \mu \frac{\partial \alpha_3}{\partial \mu} &= -2\alpha_3 \left(1 - \frac{t}{8\pi}\right) + \frac{1}{16}\alpha_1\alpha_2, \end{aligned} \tag{47}$$

and

$$\mu \frac{\partial t}{\partial \mu} = (d-2)t + \frac{C}{32M} t \sum_{i=1}^{3} \alpha_i^2. \tag{48}$$

We consider the simplified case where $\alpha_i = \alpha$ $(i = 1, 2, 3)$. In this case, the equations read

$$\begin{aligned} \mu \frac{\partial \alpha}{\partial \mu} &= -2\alpha \left(1 - \frac{t}{8\pi}\right) + \frac{1}{16}\alpha^2, \\ \mu \frac{\partial t}{\partial \mu} &= (d-2)t + \frac{C}{32} t\alpha^2. \end{aligned} \tag{49}$$

In two dimensions $d = 2$, the equations become

$$\begin{aligned} \mu \frac{\partial \alpha}{\partial \mu} &= 2\alpha v + \frac{1}{16}\alpha^2, \\ \mu \frac{\partial v}{\partial \mu} &= \frac{C}{32}\alpha^2 \end{aligned} \tag{50}$$

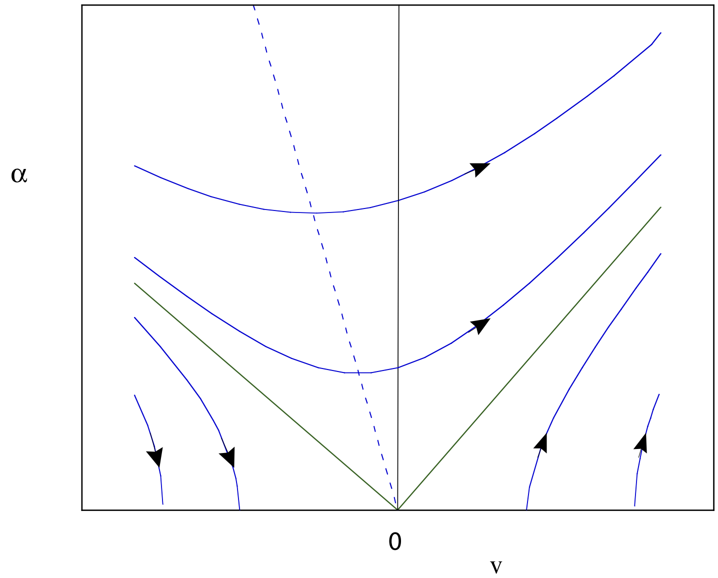

Fig. 2. Renormalization group flow as $\mu \to \infty$ in the plane of α and v.

for $t = 8\pi(1+v)$. The renormalization group flow is shown in Fig. 2 for $\alpha > 0$. The dotted line indicates $\alpha = -32v$, where $\mu\partial\alpha/\partial\mu$ vanishes. The asymptotic line as $\mu \to \infty$ is given by $\alpha \sim b_+ v$, with

$$b_+ = \frac{1}{C}\left(1 + \sqrt{1+64C}\right), \tag{51}$$

and $\alpha \sim b_- v$, with

$$b_- = \frac{1}{C}\left(1 - \sqrt{1+64C}\right). \tag{52}$$

It is apparent from Fig. 2 that there is an asymmetry between positive v and negative v. This is due to the two-vertex contribution. There is also an asymmetry between $\alpha > 0$ and $\alpha < 0$. The flow for $\alpha < 0$ is obtained just by extending straight lines into the negative α region.

4. Wilsonian renormalization group method

We investigate the renormalization of the multi-vertex sine-Gordon model by using the Wilsonian renormalization group method. We obtain the same set of equations as that in the dimensional regularization method. The only difference is that two vertices satisfying $k_i \cdot k_j \neq 0$ generate a new vertex, while k_i and k_j should satisfy $k_i \cdot k_j = \pm 1/2$ in the dimensional regularization method.

4.1. Wilsonian renormalization procedure

We write the action in the form

$$S = \int d^2x \left[\frac{1}{2}(\partial_\mu \phi)^2 + \sum_j g_j \cos(\beta k_j \cdot \phi)\right], \tag{53}$$

where $g_j = \alpha_j/t$ and $\beta = \sqrt{t}$. The field ϕ was scaled to $\beta\phi$. We reduce the cutoff Λ in the following way:

$$\Lambda \to \Lambda - d\Lambda = \Lambda - \Lambda d\ell = \Lambda e^{-d\ell}. \tag{54}$$

The scalar field $\phi = (\phi_1, \ldots, \phi_N)$ is divided into two parts as $\phi(x) = \phi_1(x) + \phi_2(x)$ with $\phi_\ell(x) = (\phi_{\ell 1}, \ldots, \phi_{\ell N})$ $(\ell = 1, 2)$, where

$$\begin{aligned} \phi_{1j}(x) &= \int_{0\le|\mathbf{p}|\le\Lambda-d\Lambda} \frac{d^2p}{(2\pi)^2} e^{i\mathbf{p}\cdot x}\phi_j(x), \\ \phi_{2j}(x) &= \int_{\Lambda-d\Lambda\le|\mathbf{p}|\le\Lambda} \frac{d^2p}{(2\pi)^2} e^{i\mathbf{p}\cdot x}\phi_j(x). \end{aligned} \tag{55}$$

The action is written as

$$\begin{aligned} S &= \int d^2x \left[\sum_{\ell=1}^{2} \frac{1}{2}(\partial_\mu \phi_\ell)^2 + \sum_j g_j \cos(\beta k_j \cdot (\phi_1 + \phi_2)) \right] \\ &= S_0(\phi_1) + S_0(\phi_2) + S_1(\phi_1, \phi_2), \end{aligned} \tag{56}$$

where S_1 indicates the potential term. Then the partition function is given by

$$Z = \int \mathcal{D}\phi e^{-S} = \int \mathcal{D}\phi_1 \exp\left(-S_0(\phi_1) + \sum_n^{\infty} \Gamma_n(\phi_1)\right), \tag{57}$$

where

$$\sum_{n=1}^{\infty} \Gamma_n(\phi_1) = \left\langle\!\!\left\langle \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n S_1^n \right\rangle\!\!\right\rangle_{\text{conn}}, \tag{58}$$

with

$$\langle\langle Q \rangle\rangle = \frac{1}{Z_2} \int \mathcal{D}\phi_2 e^{-S_0(\phi_2)} Q. \tag{59}$$

$\langle\langle\cdot\rangle\rangle_{\text{conn}}$ means keeping only connected diagrams in $\langle\langle\cdot\rangle\rangle$. Γ_n $(n = 1, 2, \ldots)$ represent contributions to the effective action.

4.2. *Lowest-order renormalization of g_j*

The lowest-order contribution $\Gamma_1 = -\langle\langle S_1 \rangle\rangle$ reads

$$\begin{aligned} \Gamma_1 &\simeq -\sum_j g_j \int d^2x \cos(\beta k_j \cdot \phi_1) \exp\left(-\frac{1}{2}\beta^2 \langle\langle (k_j \cdot \phi_2)^2 \rangle\rangle\right) \\ &= -\sum_j g_j \exp\left(-\frac{1}{2}\beta^2 G_{jd\Lambda}(0)\right) \int d^2x \cos(\beta k_j \cdot \phi_1), \end{aligned} \tag{60}$$

where the Green's function $G_{jd\Lambda}$ is defined as

$$G_{jd\Lambda}(\mathbf{x}_1 - \mathbf{x}_2) = \langle\langle \phi_{2j}(\mathbf{x}_1)\phi_{2j}(\mathbf{x}_2) \rangle\rangle = \frac{d\Lambda}{\Lambda}\frac{1}{2\pi} J_0(\Lambda|\mathbf{x}_1 - \mathbf{x}_2|), \tag{61}$$

where J_0 is the zeroth Bessel function. Up to this order, the action is renormalized to

$$\begin{aligned} S_{\Lambda-d\Lambda} &= S_0(\phi_1) - \Gamma_1 \\ &= \int d^2x \left[\frac{1}{2}(\partial_\mu \phi_1)^2 + \sum_j g_j \left(1 - \frac{\beta^2}{2} G_{jd\Lambda}(0) \right) \cos(\beta k_j \cdot \phi) \right]. \end{aligned} \tag{62}$$

We perform the following scale transformation:

$$\begin{aligned} \mathbf{x} &\to \mathbf{x}' = e^{-d\ell}\mathbf{x}, \\ \mathbf{p} &\to \mathbf{p}' = e^{d\ell}\mathbf{p}, \\ \phi_1(\mathbf{p}) &\to \tilde{\phi_1}(\mathbf{p}') = \phi_1(\mathbf{p})\zeta^{-1}, \end{aligned} \tag{63}$$

where ζ is the scaling parameter for the field ϕ_1. In the real space we have

$$\phi_1(\mathbf{x}) = \zeta e^{-2d\ell} \tilde{\phi}_1(\mathbf{x}'). \tag{64}$$

Then the effective action reads

$$S_{\Lambda-d\Lambda} = \int d^2x' \left[\zeta^2 e^{-4d\ell} \frac{1}{2} (\partial'_\mu \tilde{\phi}_1(\mathbf{x}'))^2 + \sum_j g_j e^{2d\ell} \left(1 - \frac{\beta^2}{4\pi} \frac{d\Lambda}{\Lambda} \right) \cos\left(\beta\zeta e^{-2d\ell} k_j \cdot \tilde{\phi}_1(\mathbf{x}') \right) \right]. \tag{65}$$

Here we put

$$\zeta^2 e^{-4d\ell} = 1, \tag{66}$$

so that we obtain

$$S_\Lambda = \int d^2x' \left[\frac{1}{2} (\partial'_\mu \tilde{\phi}_1(\mathbf{x}'))^2 + \sum_j g_j \left(1 + 2\frac{d\Lambda}{\Lambda} - \frac{\beta^2}{4\pi} \frac{d\Lambda}{\Lambda} \right) \cos\left(\beta k_j \cdot \tilde{\phi}_1(\mathbf{x}') \right) \right]. \tag{67}$$

This leads to the renormalized g_{Rj} and β_R as

$$g_{Rj} = g_j + \left(2 - \frac{\beta^2}{4\pi} \right) g_j \frac{d\Lambda}{\Lambda}, \tag{68}$$

$$\beta_R = \beta. \tag{69}$$

Then we have

$$\begin{aligned} \Lambda \frac{dg_j}{d\Lambda} &= \left(2 - \frac{\beta^2}{4\pi} \right) g_j, \\ \Lambda \frac{d\beta}{d\Lambda} &= 0. \end{aligned} \tag{70}$$

Since $\beta^2 = t$, these results agree with those obtained by the dimensional regularization method in two dimensions.

4.3. Multi-vertex contributions

The second-order contribution to the effective action is

$$\begin{aligned}\Gamma_2 &= \frac{1}{2}\langle\langle S_1^2\rangle\rangle_{\text{conn}}\\ &= \frac{1}{2}\sum_{ij} g_i g_j \int d^2x d^2x' \Big[\langle\langle\cos(\beta k_i\cdot\phi(\mathbf{x}))\cos(\beta k_j\cdot\phi(\mathbf{x}'))\rangle\rangle\\ &\quad - \langle\langle\cos(\beta k_i\cdot\phi(\mathbf{x}))\rangle\rangle\langle\langle\cos(\beta k_j\cdot\phi(\mathbf{x}'))\rangle\rangle\Big]. \end{aligned} \tag{71}$$

We integrate out contributions with respect to ϕ_2. For example, we use

$$\langle\langle e^{i\beta(sk_i\cdot\phi_2(\mathbf{x})+s'k_j\cdot\phi_2(\mathbf{x}'))}\rangle\rangle = \frac{1}{2}\beta^2\Big[(k_i^2+k_j^2)G_{d\Lambda}(0) + 2ss'k_i\cdot k_j G_{d\Lambda}(\mathbf{x}-\mathbf{x}')\Big], \tag{72}$$

where s and s' take ± 1. The second-order effective action Γ_2 is given as

$$\begin{aligned}\Gamma_2 = \frac{1}{4}\sum_{ij} g_i g_j \exp\left(-\beta^2 G_{d\Lambda}(0)\right)\int d^2x d^2x' \Big[&\left(e^{-\beta^2 k_i\cdot k_j G_{d\Lambda}(\mathbf{x}-\mathbf{x}')}-1\right)\cos\left(\beta(k_i\cdot\phi_1(\mathbf{x})+k_j\cdot\phi_1(\mathbf{x}'))\right)\\ &+\left(e^{\beta^2 k_i\cdot k_j G_{d\Lambda}(\mathbf{x}-\mathbf{x}')}-1\right)\cos(\beta(k_i\cdot\phi_1(\mathbf{x})-k_j\cdot\phi_1(\mathbf{x}')))\Big]. \end{aligned}\tag{73}$$

When $k_i\cdot k_j>0$, the second term grows large for $|\mathbf{x}-\mathbf{x}'|\to 0$, while the first term becomes small. When $k_i\cdot k_j<0$, the first term instead becomes large. Hence, we have

$$\begin{aligned}\Gamma_2 = \frac{1}{4}\sum_{ij} g_i g_j \exp\left(-\beta^2 G_{d\Lambda}(0)\right)\int d^2x d^2r \Big[&\left(e^{\beta^2|k_i\cdot k_j|G_{d\Lambda}(\mathbf{r})}-1\right)\cos\left(\beta(k_i\cdot\phi_1(\mathbf{x})\mp k_j\cdot\phi_1(\mathbf{x}+\mathbf{r}))\right)\Big], \end{aligned}\tag{74}$$

where $\mp$ takes $-$ when $k_i\cdot k_j>0$ and $+$ for $k_i\cdot k_j<0$. Since the integrand is large when $\mathbf{r}$ is small, Γ_2 is written as

$$\begin{aligned}\Gamma_2 &= \frac{1}{4}\sum_{ij} g_i g_j\beta^2|k_i\cdot k_j|\int d^2r G_{d\Lambda}(r)\\ &\quad\cdot\int d^2x\cos(\beta(k_i\mp k_j)\phi_1(\mathbf{x}))\left(1-\frac{\beta^2}{2}(\mathbf{r}\cdot\nabla(k_j\cdot\phi_1)^2)\right)\\ &\simeq \frac{1}{4}\sum_{ij} g_i g_j\beta^2|k_i\cdot k_j|\int d^2r G_{d\Lambda}(r)\int d^2x\cos(\beta(k_i\mp k_j)\phi_1(\mathbf{x}))\\ &\quad - \frac{1}{8}\sum_j g_j^2\beta^4\int d^2r r^2\frac{1}{2}G_{d\Lambda}(r)\int d^2x(\partial_\mu(k_j\cdot\phi_1))^2, \end{aligned}\tag{75}$$

where in the second term with the derivative of ϕ_1 we keep only the $k_i=k_j$ term since this term otherwise becomes small due to the oscillation of the cosine function. As discussed before, we use the approximation $\sum_j g_j^2(\partial_\mu(k_j\cdot\phi_1))^2\simeq\langle g_j^2\rangle C(\partial_\mu\phi_1)^2$.

Then the effective action reads

$$
\begin{aligned}
S_{\Lambda-d\Lambda} &= S_0(\phi_1) - \Gamma_1 - \Gamma_2 \\
&= \int d^2x \Bigg[\frac{1}{2}(\partial_\mu \phi_1)^2 \left(1 + \frac{A}{8}\beta^4 \langle g_j^2 \rangle \frac{d\Lambda}{\Lambda^5}\right) \\
&\quad + \sum_j g_j \left(1 - \frac{\beta^2}{2} G_{d\Lambda}(0)\right) \cos(\beta k_j \cdot \phi_1) \\
&\quad - \frac{1}{4} B \sum_{ij} g_i g_j \beta^2 |k_i \cdot k_j| \frac{d\Lambda}{\Lambda^3} \cos(\beta(k_i \mp k_j)\phi_1(\mathbf{x})) \Bigg],
\end{aligned} \tag{76}
$$

where A and B are constants defined by

$$
A = C \int_0^1 drr^3 J_0(r), \qquad B = \int_0^1 drr J_0(r). \tag{77}
$$

We perform the scale transformation in Eqs. (63) and (64), where the parameter ζ is chosen as

$$
\zeta^2 e^{-4d\ell} \left(1 + \frac{A}{8}\beta^4 \langle g_j^2 \rangle \frac{d\Lambda}{\Lambda^5}\right) = 1. \tag{78}
$$

Then the renormalized action is given by

$$
\begin{aligned}
S_\Lambda = \int d^2x \Bigg[&\frac{1}{2}(\partial_\mu \tilde{\phi}_1)^2 \\
&+ \sum_j g_j \left(1 + 2\frac{d\Lambda}{\Lambda} - \frac{\beta^2}{4\pi}\frac{d\Lambda}{\Lambda}\right) \cos(\beta\zeta e^{-2d\ell} k_j \cdot \tilde{\phi}_1(\mathbf{x})) \\
&- \frac{B}{4}\beta^2 \sum_{ij} g_i g_j |k_i \cdot k_j| \frac{d\Lambda}{\Lambda^3} \cos(\beta\zeta e^{-2d\ell}(k_i \mp k_j) \cdot \tilde{\phi}_1(\mathbf{x})) \Bigg].
\end{aligned} \tag{79}
$$

This results in the following renormalization group equations:

$$
\begin{aligned}
\Lambda \frac{d\beta}{d\Lambda} &= -\frac{A}{16\Lambda^4}\beta^5 \langle g_j^2 \rangle, \\
\Lambda \frac{dg_j}{d\Lambda} &= \left(2 - \frac{\beta^2}{4\pi}\right) g_j - \frac{B}{4\Lambda^2}\beta^2 {\sum_{i\ell}}' g_i g_\ell |k_i \cdot k_\ell|,
\end{aligned} \tag{80}
$$

where the summation is taken for k_i and k_ℓ satisfying $k_j = k_i \mp k_\ell$.

The resulting equations are consistent with those obtained using dimensional regularization. Note that the sign is different because the derivative is calculated in the descending direction $\Lambda \to \Lambda - d\Lambda$ in the Wilsonian method. In the dimensional regularization method, the summation for g_i and g_ℓ is restricted to k_i and k_ℓ that satisfy $k_i \cdot k_\ell = \pm 1/2$. In the Wilsonian method, this condition is relaxed and a new vertex is generated unless k_i and k_ℓ are orthogonal.

5. Generalized multi-vertex model

5.1. Renormalization of α_j

As shown in the evaluation of $\beta(t)$, the corrections to t are dependent on the momentum parameters $\{k_i\}$. We examine this in this section. We consider the generalized Lagrangian, given as

$$\mathcal{L} = \sum_\ell \frac{Z_\phi}{2t_\ell \mu^{2-d} Z_{t_\ell}} (\partial_\mu \phi_\ell)^2 + \sum_j \frac{\mu^d \alpha_j Z_{\alpha_j}}{t_j Z_{t_j}} \cos\left(\sqrt{Z_\phi} k_j \cdot \phi\right). \tag{81}$$

The potential term is renormalized to

$$\alpha_j Z_{\alpha_j} \exp\left(-\frac{1}{2} Z_\phi \sum_\ell k_{j\ell}^2 \langle \phi_\ell^2 \rangle\right) \cos\left(\sqrt{Z_\phi} k_j \cdot \phi\right), \tag{82}$$

where

$$\langle \phi_\ell^2 \rangle = \frac{t_\ell \mu^{2-d} Z_{t_\ell}}{Z_\phi} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m_0^2} = -\frac{t_\ell \mu^{2-d} Z_{t_\ell}}{Z_\phi} \frac{1}{\epsilon} \frac{\Omega_d}{(2\pi)^d}. \tag{83}$$

Then the correction is written as

$$\alpha_j Z_{\alpha_j} \left(1 + \frac{1}{2\epsilon} \sum_\ell k_{j\ell}^2 t_\ell \mu^{2-d} Z_{t_\ell} \frac{\Omega_s}{(2\pi)^d}\right) \cos\left(\sqrt{Z_\phi} k_j \cdot \phi\right). \tag{84}$$

This results in

$$Z_{\alpha_j} = 1 - \frac{1}{4\pi\epsilon} \sum_\ell k_{j\ell}^2 t_\ell. \tag{85}$$

Then we obtain

$$\mu \frac{\partial \alpha_j}{\partial \mu} = -2\alpha_j \left(1 - \frac{1}{8\pi} \sum_\ell k_{j\ell}^2 t_\ell\right). \tag{86}$$

The fixed point of $\{t_\ell\}$ is obtained as a zero of this equation. For an equilateral triangle where $N = 2$ and $M = 3$, we can choose $\{k_j\}$ as

$$k_1 = (1, 0), \qquad k_2 = (1/2, \sqrt{3}/2), \qquad k_3 = (-1/2, \sqrt{3}/2). \tag{87}$$

The critical value of t_ℓ is obtained as

$$t_{1c} = t_{2c} = t_{3c} = 8\pi. \tag{88}$$

For $N = 3$ we can consider a regular tetrahedron with $M = 6$, and $\{k_j\}$ are set as

$$\begin{aligned} &k_1 = (1, 0, 0),\ k_2 = (1/2, \sqrt{3}/2, 0),\ k_3 = (-1/2, \sqrt{3}/2, 0), \\ &k_4 = (1/2, 1/2\sqrt{3}, \sqrt{2/3}),\ \ k_5 = (1/2, -1/2\sqrt{3}, \sqrt{2/3}), \\ &k_6 = (0, -1/\sqrt{3}, \sqrt{2/3}). \end{aligned} \tag{89}$$

In this case, the fixed point of $\{t_\ell\}$ is also given by $t_{1c} = t_{2c} = \cdots = t_{6c} = 8\pi$.

5.2. Renormalization of t_ℓ

From the second-order perturbation appears the term that renormalizes the kinetic term, as shown in Sect. 3. We use the following approximation here:

$$\begin{aligned}\cos\left(\sqrt{Z_\phi}k_j\cdot(\phi(\mathbf{x})-\phi(\mathbf{x}+\mathbf{r}))\right)&=\cos\left(\sqrt{Z_\phi}r_\mu\partial_\mu(k_j\cdot\phi(\mathbf{x}))-\cdots\right)\\&=1-\frac{1}{2}Z_\phi\left(r_\mu\partial_\mu(k_j\cdot\phi(\mathbf{x}))\right)^2+\cdots\\&=1-\frac{1}{2}Z_\phi r_\mu r_\nu\sum_{\ell m}k_{j\ell}k_{jm}\partial_\mu\phi_\ell\partial_\nu\phi_m+\cdots.\end{aligned}\tag{90}$$

We keep the diagonal terms $(\partial_\mu\phi_\ell)^2$, and then $I_{1a}^{(2)}$ in Sect. 3 becomes

$$I_{1a}^{(2)}\simeq-\frac{1}{32\epsilon}\sum_{j\ell}\frac{\mu^{d-2}Z_\phi}{2t_jZ_{\alpha_j}}\mu^{d+2}a^4\alpha_j^2k_{j\ell}^2(\partial_\mu\phi_\ell)^2,\tag{91}$$

where we put $t_\ell=8\pi(1+v_\ell)$ and neglect terms of order v_ℓ. Then, the kinetic term is renormalized into

$$\sum_\ell\frac{Z_\phi}{2t_\ell\mu^{2-d}Z_{t_\ell}}\left[1-\frac{1}{32\epsilon}\mu^{d+2}a^4\sum_j\alpha_j^2k_{j\ell}^2\right](\partial_\mu\phi_\ell)^2.\tag{92}$$

This leads to

$$Z_{t_\ell}=1-\frac{1}{32\epsilon}\mu^{d+2}a^4\sum_j\alpha_j^2k_{j\ell}^2.\tag{93}$$

Then we obtain

$$\mu\frac{\partial t_\ell}{\partial\mu}=(d-2)t_\ell+\frac{1}{32}t_\ell\sum_j\alpha_j^2k_{j\ell}^2.\tag{94}$$

For $N=2$ and $M=3$ we use $\{k_j\}$ for an equilateral triangle; the equations for t_1 and t_2 read

$$\mu\frac{\partial t_1}{\partial\mu}=(d-2)t_1+\frac{1}{32}t_1\left(\alpha_1^2+\frac{1}{4}\alpha_2^2+\frac{1}{4}\alpha_3^2\right),\tag{95}$$

$$\mu\frac{\partial t_2}{\partial\mu}=(d-2)t_2+\frac{1}{32}t_2\left(\frac{3}{4}\alpha_2^2+\frac{3}{4}\alpha_3^2\right).\tag{96}$$

This is the result for the generalized multi-vertex sine-Gordon model. The qualitative property is the same as that obtained in Sect. 3. When $\alpha\equiv\alpha_1\sim\alpha_2\sim\alpha_3$, we have

$$\mu\frac{\partial t_1}{\partial\mu}=(d-2)t_1+\frac{C}{32}t_1\alpha^2,\tag{97}$$

with $C=3/2$. This agrees with the previous result.

5.3. Multi-vertex contribution to α_j

For the generalized model, $I_{1b}^{(2)}$ in Sect. 3 becomes

$$I_{1b}^{(2)} = -\frac{1}{4}\sum_{j\neq i}\frac{\mu^d\alpha_j Z_{\alpha_j}}{t_j Z_{t_j}}\frac{\mu^d\alpha_j Z_{\alpha_i}}{t_j Z_{t_i}}\times$$
$$\int d^dxd^dr\exp\Big[-\frac{Z_\phi}{2}\sum_\ell(k_{j\ell}^2+k_{i\ell}^2)\langle\phi_\ell^2\rangle+\sum_\ell k_{j\ell}k_{i\ell}t_\ell\mu^{2-d}Z_{t_\ell}\frac{1}{2\pi}K_0(m_0r)\Big]$$
$$\cdot\cos\Big(\sqrt{Z_\phi}(k_i\cdot\phi(x)-k_j\cdot\phi(x+\mathbf{r}))\Big). \tag{98}$$

The integral with respect to r becomes

$$J_{ij} := \int d^dr\exp\left(\sum_\ell k_{j\ell}k_{i\ell}t_\ell\mu^{2-d}Z_{t_\ell}\frac{1}{2\pi}K_0(m_0r)\right)$$
$$\simeq \Omega_d\int_0^\infty drr^{d-1}\left(\frac{1}{cm_0^2(r^2+a^2)}\right)^{\sum_\ell k_{j\ell}k_{i\ell}t_\ell/4\pi}. \tag{99}$$

We consider the region near the fixed point $t_\ell = 8\pi(1+\nu_\ell)$, where J_{ij} is estimated as

$$J_{ij} = -\Omega_d(cm_0^2)^{-1}\frac{1}{\epsilon}+O(\nu) \tag{100}$$

when $k_i\cdot k_j = 1/2$. This indicates that

$$I_{1b}^{(2)} \simeq \frac{1}{16\epsilon}\frac{cm_0^2}{8\pi}\sum_{i\neq j}\alpha_j\alpha_i\mu^{2d}a^4\int d^dx\cos\Big(\sqrt{Z_\phi}(k_i-k_j)\cdot\phi(x)\Big). \tag{101}$$

The potential term with two-vertex correction is obtained as

$$\sum_\ell\frac{\mu^d\alpha_\ell Z_{\alpha_\ell}}{t_\ell Z_{t_\ell}}\Bigg[1+\frac{1}{4\pi\epsilon}\sum_m k_{\ell m}^2t_m+\frac{1}{16\epsilon}{\sum_{i\neq j}}'\frac{\alpha_j\alpha_i}{\alpha_\ell}\mu^dcm_0^2a^4\Bigg]\cos\Big(\sqrt{Z_\phi}k_\ell\cdot\phi\Big), \tag{102}$$

where $\sum'_{i\neq j}$ indicates summation under the condition that $k_i\pm k_j = k_\ell$. Then we choose Z_{α_ℓ} as

$$Z_{\alpha_\ell} = 1-\frac{1}{4\pi\epsilon}\sum_m k_{\ell m}^2t_m-\frac{1}{16\epsilon}{\sum_{i\neq j}}'\frac{\alpha_i\alpha_j}{\alpha_\ell}\mu^dcm_0^2a^4. \tag{103}$$

The beta function up to the second order of α is given as

$$\mu\frac{\partial\alpha_\ell}{\partial\mu} = -2\alpha_\ell\left(1-\frac{1}{8\pi}\sum_m k_{\ell m}^2t_m\right)+\frac{1}{16}{\sum_{i\neq j}}'\alpha_i\alpha_j, \tag{104}$$

where we set $cm_0^2a^2 = 1$.

6. Summary

We have investigated the multi-vertex sine-Gordon model on the basis of renormalization group theory. We employed the dimensional regularization method and the Wilsonian renormalization group method; the two results are consistent with each other. The generalized sine-Gordon model contains multiple cosine (vertex) potentials labelled by momentum parameters $\{k_j\}_{j=1,\ldots,M}$. The vertex–vertex scattering amplitude is given by the tachyon scattering amplitude. A new vertex k_ℓ is generated from two vertex interactions k_i and k_j, and they are closed when momentum parameters $\{k_j\}$ satisfy the triangle condition that $k_i \pm k_j = k_\ell$. When k_i and k_j are orthogonal, a new vertex is not generated. The condition $k_i \cdot k_j = \pm 1/2$ is required in the dimensional regularization method.

For a two-component scalar field ($N = 2$), $\{k_j\}$ should form a triangle (Wilson method) or an equilateral triangle (dimensional regularization) for $M = 3$. For a three-component scalar field ($N = 3$), a regular tetrahedron forms a closed system for $M = 6$. For these structures, the fixed point of $\{t_j\}$ is given by $t_1 = t_2 = \cdots = t_M$. A regular octahedron is also possible where there are six independent k_j and thus $M = 6$. For an equilateral triangle, regular tetrahedron, and regular octahedron, we have $\sum_j k_{j\ell}^2 = C(M)$ for $\ell = 1, \ldots, N$, where we impose the normalization $\sum_\ell k_{j\ell}^2 = 1$. We expect that there exist crystal structures in higher dimensions $N \geq 3$ satisfying $\sum_j k_{j\ell}^2 = \text{const.}$ for any ℓ.

The beta function of α_ℓ is generalized to include the product $\alpha_i\alpha_j$ for which $k_i \pm k_j = k_\ell$ is satisfied. This term is a non-trivial contribution compared to the conventional sine-Gordon model. The beta function of t_ℓ also has contributions proportional to α_j^2. These terms are positive and thus do not change the renormalization group flow of t_ℓ. The additional terms to $\beta(\alpha_\ell)$ change the flow of (α_ℓ, t_j) qualitatively.

Acknowledgements

This work was supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan (Grant No. 17K05559).

Funding

Open Access funding: SCOAP^3.

References

[1] S. Coleman, Phys. Rev. D **11**, 2088 (1975).
[2] R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **11**, 3424 (1975).
[3] J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. B **16**, 1217 (1977); **17**, 1477 (1978) [erratum].
[4] S. Samuel, Phys. Rev. D **18**, 1916 (1978).
[5] A. B. Zamolodchikov, and Al. B. Zamolodchikov, Ann. Phys. **120**, 253 (1979).
[6] D. J. Amit, Y. Y. Goldschmidt, and S. Grinstein, J. Phys. A: Math. Gen. **13**, 585 (1980).
[7] P. B. Wiegmann, J. Phys. C: Solid State Phys. **11**, 1583 (1978).
[8] A. A. Tseytlin, Phys. Lett. B **241**, 233 (1990).
[9] J. Balog and Á. Hegedus, J. Phys. A: Math. Gen. **33**, 6543 (2000).
[10] R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982).
[11] N. S. Manton and P. Sutcliffe, *Topological Solitons* (Cambridge University Press, Cambridge, 2004).
[12] S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, 1985).
[13] E. C. Marino, *Quantum Field Theory Approach to Condensed Matter Physics* (Cambridge University Press, Cambridge, 2017).
[14] E. Weinberg, *Classical Solutions in Quantum Field Theory* (Cambridge University Press, Cambridge, 2015).
[15] I. Herbut, *A Modern Approach to Critical Phenomena* (Cambridge University Press, Cambridge, 2007).
[16] I. Nándori, U. D. Jentschura, K. Sailer, and G. Soff, Phys. Rev. D **69**, 025004 (2004).
[17] S. Nagy, I. Nándori, J. Polonyi, and K. Sailer, Phys. Rev. Lett. **102**, 241603 (2009).
[18] J. José, Phys. Rev. D **14**, 2826 (1976).

[19] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, Oxford, 1989).
[20] J. M. Kosterlitz and D. J. Thouless, J. Phys. C: Solid State Phys. **6**, 1181 (1973).
[21] J. M. Kosterlitz, J. Phys. C: Solid State Phys. **7**, 1046 (1974).
[22] V. L. Golo and A. M. Perelomov, Phys. Lett. B **79**, 112 (1978).
[23] E. Brézin, C. Itzykson, J. Zinn-Justin, and J.-B. Zuber, Phys. Lett. B **82**, 442 (1979).
[24] T. Yanagisawa, Europhys. Lett. **113**, 41001 (2016).
[25] J. Wess and B. Zumino, Phys. Lett. B **37**, 95 (1971).
[26] E. Witten, Nucl. Phys. B **223**, 422 (1983).
[27] E. Witten, Commun. Math. Phys. **92**, 455 (1984).
[28] S. P. Novikov, Russian Mathematical Surveys **37**, 1 (1982).
[29] I. Nándori, S. Nagy, K. Sailer, and A. Trombettoni, Phys. Rev. D **80**, 025008 (2009).
[30] G. Delfino and G. Mussardo, Nucl. Phys. B **516**, 675 (1998).
[31] T. Yanagisawa, Prog. Theor. Exp. Phys. **2019**, 023A01 (2019).
[32] P. Oak and B. Sathiapalan, Phys. Rev. D **99**, 046009 (2019).
[33] S. R. Das and B. Sathiapalan, Phys. Rev. Lett. **56**, 2664 (1986).
[34] G. 'tHooft and M. Veltman, Nucl. Phys. B **44**, 189 (1972).
[35] D. Gross, *Methods in Field Theory*, Les Houches Lecture Notes, eds. R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976).
[36] T. Yanagisawa, Renormalization group theory of effective field theory models in low dimensions, in *Recent Studies in Perturbation Theory*, ed. D. I. Uzunov (InTechOpen, London, 2017) [arXiv:1804.02845 [cond-mat.stat-mech]] [Search INSPIRE].
[37] M. Kaku, *Introduction to Superstrings and M-theory* (Springer, Berlin, 1988).
[38] Z. Koba and H. B. Nielsen, Nucl. Phys. B **10**, 633 (1969).