A General (Category Theory) Principle for General Intelligence: Duality (Adjointness)

Steven Phillips(✉)

Mathematical Neuroinformatics Group, Human Informatics Research Institute,
National Institute of Advanced Industrial Science and Technology (AIST),
Tsukuba, Ibaraki, Japan
steve@ni.aist.go.jp

Abstract. Artificial General Intelligence (AGI) seeks theories, models and techniques to endow machines with the kinds of intellectual abilities exemplified by humans. Yet, the predominant instance-driven approach in AI appears antithetical to this goal. This situation raises a question: What (if any) general principles underlie general intelligence? We approach this question from a (mathematical) category theory perspective as a continuation of a categorical approach to other properties of human cognition. The proposal pursued here is adjoint functors as a universal (systematic) basis for trading the costs/benefits that accompany physical systems interacting intelligently with their environment.

Keywords: Systematicity · Compositionality · category theory · Functor · adjunction · Intelligence · Raven Progressive Matrices

1 Introduction

The purview of Artificial *General* Intelligence (AGI) is the development of theories, models and techniques for the endowment of machines with intellectual capabilities that *generalize* to a variety of novel situations. This characterization, however, belies important questions about what we mean by intelligence and generalize. In the absence of precise criteria, researchers look to the archetype of general intelligence, human cognition, for examples of model behaviour [19].

Such (behaviourist/operationalist) approaches afford clear criteria to compare methods, but there are some significant drawbacks. Firstly, complex behaviour can be realized in more than one way. A machine that requires many more training examples to achieve a comparable level of human performance on a complex intellectual activity (e.g., chess) may not capture essential properties of general intelligence [13]. Secondly, humans also make (logically) irrational decisions [12]. Failures of logical reasoning, however, do not warrant rejecting human cognition as an example of general intelligence. So, specific behaviours may provide neither necessary nor sufficient criteria for general intelligence.

This problematic state of affairs raises an important question: What (if any) general principles underlie general intelligence? Discerning principles for cognition is a concern of cognitive scientists when comparing/contrasting mental

T. Everitt et al. (Eds.): AGI 2017, LNAI 10414, pp. 57–66, 2017.
DOI: 10.1007/978-3-319-63703-7_6

capacity across cohorts (e.g., age groups, or species). A typical recourse is to look at relationships between mental capacities, rather than individual behaviours [10]. In the remainder of this introduction, we recall one such relationship that motivates our approach to AGI, which is presented in the subsequent sections.

1.1 Systematicity, Generalization and Categorical Universality

The so-called *cognitive revolution* in psychology was a shift in focus from behaviour to the underlying structures that generate it, or more pointedly, a shift towards the (structural) relations between the underlying cognitive processes that cause the structural relations between behaviours generated [3]. An example is the *systematicity* property of cognition. Systematicity is when having a capacity for some cognitive ability implies having a capacity for a structurally-related ability [8]. An example is having the capacity to understand the expression *John loves Mary* if and only if having the capacity to understand *Mary loves John*. These two capacities are related by the common *loves* relation. Systematicity, in general, is an equivalence relation over cognitive capacities, which need not be confined to language [15]—a kind of generalization over cognitive abilities.

The systematicity problem is to explain *why* cognition is organized into particular groups of cognitive capacities [8]. Although this problem was articulated three decades ago, consensus on a solution remains elusive (see [4] for a recent reappraisal). Cognitive scientists generally agree that systematicity depends on processing common structure, though they may disagree on the nature of those processes, e.g., symbolic [8], or subsymbolic [20]. However, the sticking point is over a specification for the (necessary and sufficient) conditions from which systematicity follows: the *why* not just the *how* of systematicity [1,8]. Central to (ordinary) *category theory* [14] is the formal concept of *universal construction*: necessary and sufficient conditions relating collections of mathematically structured objects. In this sense (of necessity and sufficiency) one can regard category theory as a *theory of structure*, which should make category theory well-placed to provide an explanation for the why of systematicity [17].

A category consists of a collection of *objects*, a collection of *morphisms* (also called *arrows*, or *maps*), and a *composition operation* for composing morphisms. In the context of cognition, morphisms may be regarded as cognitive processes that map between objects that are sets of cognitive states. A *universal morphism* (universal construction) is a morphism that is common to a collection of morphisms, hence its relevance to an explanation for systematicity [17].

1.2 Cost/Benefit Cognition: Dual-Routes and Duality

If cognition is supposed to be systematic, then why are there failures of systematicity? Cognitive systems are physical systems, hence resource sensitive. So, alternative ways of realizing task goals may trade one kind of resource for another. For example, parallel computation typically involves more memory (space) but less time than serial computation; faster response is typically accompanied by lower accuracy. We hypothesized that failures of systematicity arise from a cost/benefit trade-off associated with employing a universal construction, and an experiment designed to manipulate the cost of computing a task *with* versus *without* a universal construction provided support for this hypothesis [16].

Characterizations of cognition as dual-process (route) abound in psychology: e.g., fast versus slow, domain-specific versus domain-general, resilient versus sensitive to working memory load, and associative versus relational [7,10,12]. Although identifying such distinctions are important, they do not explain why cognition appears this way. Our study [16] suggested that failures of systematicity are themselves systematically related. Since the categorical explanation says that a universal construction underlies each and every instance of systematicity, we propose that another kind of universal construction, called an *adjunction*, underlies cognitive dual-routes and general intelligence.

An adjunction can be considered as a collection of universal morphisms for the opposing constructions as dual-routes. Each collection affords a systematic alternative path that realizes a cost/benefit trade-off. General intelligence involves the effective exploitation of this trade-off. The link from dual-route to adjunction is formally illustrated using a familiar example of dual from elementary algebra, in Sect. 2, which also serves as an aid to understanding the basic category theory that follows for application to cognition and general intelligence, in Sect. 3. This general principle for AGI is discussed in Sect. 4.

2 Categorical Dual (Adjunction): An Elementary Example

Computing with very small or large numbers creates precision errors when results exceed a machine's representational capacity. These computational "potholes" are avoided by taking a dual route, which is illustrated using the following equations relating addition to multiplication:

$$a \times b = e^{\log a + \log b} \quad \text{and} \tag{1}$$

$$a + b = \log(e^a \times e^b)\,, \tag{2}$$

which show that one can be computed in terms of the other.

Definition 1 (Category). *A category* $\mathbf{C}$ *consists of*

- *a collection of objects,* $\mathcal{O}(\mathbf{C}) = \{A, B, \dots\}$*,*
- *a collection of morphisms,* $\mathcal{M}(\mathbf{C}) = \{f, g, \dots\}$*—*$f : A \to B$ *indicates* A *as the domain and* B *as the codomain of* f*, and* $\mathrm{Hom}_{\mathbf{C}}(A, B)$ *as the collection of morphisms from* A *to* B *in* $\mathbf{C}$*—including the morphism* $1_A : A \to A$ *for every object* $A \in \mathcal{O}(\mathbf{C})$*, called the identity morphism at* A*, and*
- *a composition operation,* $\circ$*, that sends a pair of morphisms* $f : A \to B$ *and* $g : B \to C$ *to the composite morphism* $g \circ f : A \to C$*,*

that together satisfy

- *identity:* $f \circ 1_A = f = 1_B \circ f$ *for every* $f \in \mathcal{M}(\mathbf{C})$*, and*
- *associativity:* $h \circ (g \circ f) = (h \circ g) \circ f$ *for every triple of compatible morphisms* $f, g, h \in \mathcal{M}(\mathbf{C})$*: the codomain of* f *is the domain of* g*; likewise for* g *and* h*.*

Example 1 (Set). *The category* **Set** *has sets for objects, functions for morphisms, and composition is composition of functions:* $g \circ f(a) = g(f(a))$*. Identity morphisms are identity functions:* $1_A : a \mapsto a$*.*

Example 2 (Monoid). *A monoid is a set* M *with a binary operation* $\cdot$ *and an identity element* $e \in M$ *such that* $a \cdot e = a = e \cdot a$ *for every element* $a \in M$*. Every monoid* $(M, \cdot, e)$ *is a one-object category whose morphisms are the elements of* M*, with* e *as the identity morphism, and composition is the monoid operation. The set of real numbers* $\mathbb{R}$ *under addition and multiplication are the monoids* $(\mathbb{R}, +, 0)$ *and* $(\mathbb{R}, \times, 1)$ *and therefore categories. For instance, the composition of morphisms* $2 : * \rightarrow *$ *and* $3 : * \rightarrow *$ *is the morphism* $3 \circ 2 = 5 : * \rightarrow *$*, which corresponds to the addition of their corresponding numbers,* $2 + 3 = 5$*.*

Remark 1. *Category* $\mathbf{C^{op}}$ *is opposite to* $\mathbf{C}$*, which is obtained by morphism reversal: morphism* $f : A \rightarrow B$ *in* $\mathbf{C}$ *is* $f^{op} : B \rightarrow A$ *in* $\mathbf{C^{op}}$*. A dual (e.g., coproduct) in* $\mathbf{C}$ *is just the primal (product) in* $\mathbf{C^{op}}$*.*

Definition 2 (Functor). *A functor* $F : \mathbf{C} \rightarrow \mathbf{D}$ *is a map from category* $\mathbf{C}$ *to category* $\mathbf{D}$ *sending each object* A *and morphism* $f : A \rightarrow B$ *in* $\mathbf{C}$ *to (respectively) the object* $F(A)$ *and the morphism* $F(f) : F(A) \rightarrow F(B)$ *in* $\mathbf{D}$ *such that*

- *identity:* $F(1_A) = 1_{F(A)}$ *for every object* $A \in \mathcal{O}(\mathbf{C})$*, and*
- *compositionality:* $F(g \circ_{\mathbf{C}} f) = F(g) \circ_{\mathbf{D}} F(f)$ *for every pair of compatible morphisms* $f, g \in \mathcal{M}(\mathbf{C})$*.*

Example 3 (Monoid homomorphism). *A monoid homomorphism is a map* $h : (M, \cdot, e) \rightarrow (N, \star, e')$ *such that* $h(e) = e'$ *and* $h(a \cdot b) = h(a) \star h(b)$ *for all* $a, b \in M$*. Every monoid homomorphism is a functor. For instance, the exponential function* $\exp : a \mapsto e^a$ *is a monoid homomorphism, since* $e^0 = 1$ *and* $e^{a+b} = e^a \times e^b$*, and therefore a functor. Likewise, the log function defined over* $\mathbb{R} - \{0\}$ *(denoted* $\mathbb{R}_0$*) is a functor, since* $\log(1) = 0$ *and* $\log(a \times b) = \log(a) + \log(b)$*.*

Remark 2. *A functor* $F : \mathbf{C^{op}} \rightarrow \mathbf{D}$ *is called a contravariant functor. Contravariant functor* $I^{op} : \mathbf{C^{op}} \rightarrow \mathbf{C}$ *sends* f^{op} *to* f*.*

The following definition is needed before defining adjunction. We delay giving examples until the next section, because the natural transformations associated with the current example of a dual are trivial (i.e. identities).

Definition 3 (Natural transformation, isomorphism). *A natural transformation* η *from a functor* $F : \mathbf{C} \rightarrow \mathbf{D}$ *to a functor* $G : \mathbf{C} \rightarrow \mathbf{D}$*, written* $\eta : F \dot{\rightarrow} G$*, is a family of* $\boldsymbol{D}$*-morphisms* $\{\eta_A : F(A) \rightarrow G(A) | A \in \mathcal{O}(\mathbf{C})\}$ *such that* $G(f) \circ \eta_A = \eta_B \circ F(f)$ *for each morphism* $f : A \rightarrow B$ *in* $\mathbf{C}$*. A natural isomorphism is a natural transformation where every* η_A *is an isomorphism, i.e. a morphism that has a (left/right) inverse.*

Remark 3. *A natural isomorphism is indicated by the following diagram:*

$$\begin{array}{ccc} F(A) & \underset{\eta_A^{-1}}{\overset{\eta_A}{\rightleftarrows}} & G(A) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle G(f)} \\ F(B) & \underset{\eta_B^{-1}}{\overset{\eta_B}{\rightleftarrows}} & G(B)\,, \end{array} \tag{3}$$

which yields two identities:

$$F(f) = \eta_B^{-1} \circ G(f) \circ \eta_A \quad \textit{and} \tag{4}$$

$$G(f) = \eta_B \circ F(f) \circ \eta_A^{-1}\,, \tag{5}$$

hence their importance in exploiting dual-routes.

Definition 4 (Adjunction). *An adjunction* $(F, G, \eta, \epsilon) : \mathbf{C} \rightharpoonup \mathbf{D}$ *consists of functors* $F : \mathbf{C} \to \mathbf{D}$ *and* $G : \mathbf{D} \to \mathbf{C}$*, and natural transformations* $\eta : 1_{\mathbf{C}} \dot{\to} G \circ F$ *and* $\epsilon : F \circ G \dot{\to} 1_{\mathbf{D}}$ *satisfying* $G\epsilon \circ \eta G = 1_G$ *and* $\epsilon F \circ F\eta = 1_F$*, where* 1_F *and* 1_G *are identity natural transformations (on* F *and* G*).* F *is the called the left adjoint of* G*, and* G *is called the right adjoint of* F*, written* $F \dashv G$*; natural transformations* η *and* ϵ *are called (respectively) the unit and counit.*

Remark 4. *Definition 4 induces equalities* $f = G(g) \circ \eta_A$ *and* $g = \epsilon_B \circ F(f)$*, which are shown by the following diagrams:*

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GF(A) \\ & {\scriptstyle f}\searrow & \big\downarrow{\scriptstyle G(g)} \\ & & G(B) \end{array} \qquad \begin{array}{c} F(A) \\ \big\downarrow{\scriptstyle g} \\ B \end{array} \qquad \begin{array}{c} A \\ {\scriptstyle f}\big\downarrow \\ G(B) \end{array} \qquad \begin{array}{ccc} F(A) & & \\ {\scriptstyle F(f)}\big\downarrow & \searrow{\scriptstyle g} & \\ FG(B) & \xrightarrow[\epsilon_B]{} & B\,. \end{array} \tag{6}$$

Dashed arrows indicate uniqueness. The pair $(F(A), \eta_A)$ *is the universal morphism from* A *to* F*; the pair* $(G(B), \epsilon_B)$ *is the universal morphism from* G *to* B*. In other words, every morphism* f *factors through* η_A*; every morphism* g *factors through* ϵ_B*, hence the importance of universal morphisms to systematicity.*

Remark 5. *Derived hom-functors* $\mathrm{Hom}(F-, -), \mathrm{Hom}(-, G-) : \mathbf{C^{op}} \times \mathbf{D} \to \mathbf{Set}$ *and natural isomorphism* $\phi : \mathrm{Hom}(F-, -) \dot{\to} \mathrm{Hom}(-, G-) : \psi$ *(see [14]) are indicated by the following diagram:*

$$\begin{array}{ccccc} (A, B) & \quad & \mathrm{Hom}_{\mathbf{D}}(F(A), B) & \underset{\psi_{A,B}}{\overset{\phi_{A,B}}{\rightleftarrows}} & \mathrm{Hom}_{\mathbf{C}}(A, G(B)) \\ {\scriptstyle (h,k)}\big\downarrow & & {\scriptstyle \mathrm{Hom}(F(h),k)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}(h,G(k))} \\ (A', B') & & \mathrm{Hom}_{\mathbf{D}}(F(A'), B') & \underset{\psi_{A',B'}}{\overset{\phi_{A',B'}}{\rightleftarrows}} & \mathrm{Hom}_{\mathbf{C}}(A', G(B'))\,, \end{array} \tag{7}$$

hence the importance of adjunctions to duality and dual-routes.

steve@ni.aist.go.jp

Example 4 **(**exp ⊣ log**).** *Setting F and G in diagram 7 to functors* exp *and* log*, hence* $\mathrm{Hom}_{\mathbf{D}}(F(A), B) = \mathbb{R}_0$ *and* $\mathrm{Hom}_{\mathbf{C}}(A, G(B)) = \mathbb{R}$*, we have for* (h, k) *set to* $(0, b)$ *and* (h, k) *to* $(a, 1)$ *the following (respectively, left and right) diagrams:*

$$\begin{array}{ccc} (*,*) & \mathbb{R}_0 \xrightarrow{\log(-)} \mathbb{R} \\ {\scriptstyle (0,b)}\downarrow & {\scriptstyle \times b}\downarrow \qquad \downarrow{\scriptstyle +\log b} \\ (*,*) & \mathbb{R}_0 \xleftarrow[e^{(-)}]{} \mathbb{R} \end{array} \qquad \begin{array}{ccc} (*,*) & \mathbb{R}_0 \xleftarrow{e^{(-)}} \mathbb{R} \\ {\scriptstyle (a,1)}\downarrow & {\scriptstyle e^a \times}\downarrow \qquad \downarrow{\scriptstyle a+} \\ (*,*) & \mathbb{R}_0 \xrightarrow[\log(-)]{} \mathbb{R}\,. \end{array} \tag{8}$$

For all $a \in \mathbb{R}_0$, traversal of the left square recovers Eq. 1; for all $b \in \mathbb{R}$, traversal of the right square recovers Eq. 2 (cf. Eqs. 4 and 5).

Remark 6. *Functions/functors* log *and* exp *are mutual inverses, hence isomorphisms. Every isomorphic functor and its inverse form an adjunction, but every adjoint functor is not an isomorphism (see, e.g., next section). One can think of adjoints as conceptual though not necessarily actual inverses.*

3 Cognitive Dual-Routes and Adjoints

With the formal concept of adjunction at hand, we present two examples of how adjunctions underlie cognitive dual-routes. Both examples involve categorical products, which relate cognitive development across reasoning tasks [18].

3.1 Stimulus-Response

To examine a potential cost/benefit trade-off associated with categorical products, subjects were tested on a stimulus-response task involving a product of two maps: a character-to-colour map *char2colour* : *Char* → *Colour* and a character-to-shape map *char2shape* : *Char* → *Shape*, e.g., (G, P) ↦ (red, ♣), (P, K) ↦ (blue, ▼) [16]. Subjects could learn each task as a single map of pairs (n^2), or as a pair of maps between singletons ($2n$). The former alternative does not afford generalization, as each pair is interpreted as a unique, indivisible element; the latter alternative affords generalization after inducing the component maps. The map learned depended on set size: stimulus-response associations were learned wholistically when the number of mappings was small, but componentially when the number of mappings was large, and this difference depended on the order of learning [16]. Here, we show that the categorical basis for this duality is the adjoint relationship between *diagonal* and *product* functors.

Definition 5 (Diagonal, product functor). *The diagonal functor* $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}; A \mapsto (A, A), f \mapsto (f, f)$ *sends each object and morphism to their pairs. The product functor* $\Pi : \mathbf{C} \times \mathbf{C} \to \mathbf{C}; (A, B) \mapsto A \times B, (f, g) \mapsto f \times g$ *sends pairs of objects and morphisms to their categorical products.*

Remark 7. *The categorical product in* **Set** *is the Cartesian product.*

Example 5 **($\Delta \dashv \Pi$).** *Diagonal and product functors form an adjoint pair. The natural transformations are:* $\langle 1, 1\rangle : 1_{\mathbf{C}} \dot{\rightarrow} \Pi \circ \Delta$ *and* $(\pi_1, \pi_2) : \Delta \circ \Pi \dot{\rightarrow} 1_{\mathbf{C}\times\mathbf{C}}$. *In* **Set***,* π_1 *and* π_2 *are projections, i.e.* $\pi_1 : (a, b) \mapsto a$*, and* $\pi_2 : (a, b) \mapsto b$*. Instantiating* F *and* G *in diagram 7 as* Δ *and* Π *over* **Set** *yields*

$$\begin{array}{ccc} (A,B) & \xleftarrow{\ \phi\ } & A \times B \\ {\scriptstyle (f,g)}\downarrow & & \downarrow{\scriptstyle f\times g} \\ (A',B') & \xrightarrow[\ \psi\ ]{} & A' \times B' \end{array} \qquad \begin{array}{ccc} (a,b) & \xleftarrow{\ \phi\ } & \langle a,b\rangle \\ {\scriptstyle (f,g)}\downarrow & & \downarrow{\scriptstyle f\times g} \\ (f(a),g(b)) & \xmapsto[\ \psi\ ]{} & \langle f(a),g(b)\rangle\,. \end{array} \tag{9}$$

For the stimulus-response task, A and B (diagram 9) correspond to *Char*, and A' and B' to *Colour* and *Shape*. The dual-route realized by the adjunction trades the cost of maintaining a pair of maps (left vertical arrows in each square) with the benefit on only needing about $2n$ training examples for correct response prediction on all n^2 of the single product map (right vertical arrows).

3.2 A Measure of Intelligence: Raven Progressive Matrices

Raven Progressive Matrices (RPM) is an inference task. Subjects are presented with a 3×3 matrix of stimuli, whose bottom-right cell is empty, and an array of choice stimuli from which they choose the stimulus that belongs in the empty cell. Examples are shown in Fig. 1, with stimuli varying along one (number) or two (number, shape) dimensions. Various factors influence the difficulty of RPM, such as recognizing the relevant relations to infer the missing attributes for the row/column [5], and the number of such variable relations [6, 21]. Dimensionality pertains to (unary/binary) products (see [18] for the relationship between dimensionality, product arity and difficulty for the closely related *matrix completion* task), hence the aforementioned diagonal-product adjunction, albeit for particular algebras instead of just sets. So, here, we focus on the missing attribute aspect of RPM, as involving another instance of an adjunction.

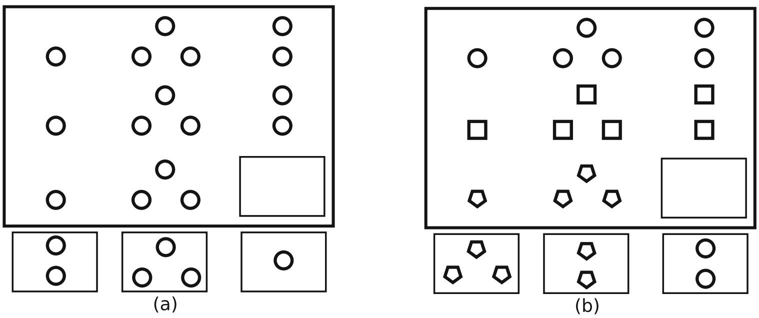

Fig. 1. RPM-like examples with (a) one and (b) two dimensions of variable relations.

The adjunction involves constructing a *free object*. Typically, the left adjoint is a *free* functor that sends each set to the free algebraic structure (e.g., monoid, group, etc.) on that set. The right adjoint is the associated *forgetful* functor that sends each algebraic structure to its underlying set, forgetting the algebraic operations. For example, the free monoid on the set (alphabet) A is the monoid $(A^*, \cdot, \varepsilon)$ consisting of the set of "words" A^* (i.e. strings of 0 or more characters $a \in A$) composed from the concatenation operation $\cdot$, where ε is the empty (length zero) word. The universal construction is shown in diagram 10 (left), where *len* is the monoid homomorphism returning word length, ι is the universal (initial) morphism, and 1 is the constant function assigning 1 to every alphabetic character. Initial morphism ι is an injection of generators $a \in A$; equivalently, the completion of word set A^* from alphabet A.

$$\begin{array}{ccc} A & \xrightarrow{\iota} & A^* \\ & \searrow^{1} & \downarrow{\scriptstyle len} \\ & & \mathbb{N} \end{array} \qquad \begin{array}{c} (A^*, \cdot, \varepsilon) \\ \downarrow{\scriptstyle len} \\ (\mathbb{N}, +, 0) \end{array} \qquad \begin{array}{ccc} \{a, b\} & \xrightarrow{\iota} & \{a, b, c\} \\ & \searrow^{f} & \downarrow{\scriptstyle g} \\ & & G \end{array} \qquad \begin{array}{c} (\{a, b, c\}, \cdot) \\ \downarrow{\scriptstyle g} \\ (G, \star)\,. \end{array} \tag{10}$$

For RPM, each row/column constitutes a *semigroupoid* (partial monoid with identity unneeded). The missing feature (e.g., shape) is obtained from the initial morphism as the completion of the two given features (circle, square) to obtain the other feature (pentagon). The initial morphism is the completion of the two-element set $\{a, b\}$ to the three-element set $\{a, b, c\}$, diagram 10 (right). The semigroupoid formalizes the notion of obtaining the missing element c from the given elements a and b, i.e. $a \cdot b = c$, where $\cdot$ is the semigroupoid operation.[1] There is a speed accuracy trade-off with regard to products: considering a single dimension is faster but less accurate, e.g., neither shape nor number uniquely identifies the target (two pentagons) in Fig. 1(b), see also Discussion.

4 Discussion

We have looked at three examples of adjunctions as the basis of dual-routes and cost/benefit trade-offs. Given the diversity of what one may regard as general intellectual behaviour, claims of a general principle from so few examples may seem premature. In what sense, then, are adjunctions justifiably a general principle for general intelligence? In the remainder of this section, we step back from the formal details to discuss some broader conceptual motivations.

The conceptual connections between general intelligence and adjunction are the following. General intelligence is a product of cognition, cognitive systems are physical systems, physical systems interact with their environment by exchange of energy (information), and this interaction (adjunction) induces a dual-route.

[1] Equivalently, the missing element is obtained from the underlying graph of the *free semicategory* (category with identity arrows unneeded) on the graph consisting of the connected edges a and b: the missing element is the edge $c = ab$.

Cost/benefit can be regarded as a duality between system and environment: cost is the expenditure of system resources on the environment, and benefit is uptake of environmental resources by the system. Formally, we have regarded this dual relation as adjunction, and choice depends on which route is more cost effective.

We presented three examples of how cost/benefit trade-off may arise from adjunction. Directly adding/multiplying very small or large numbers effectively has large cost when representational capacity is exceeded: *enlarge is dual to compress*. Directly inferring a response to a novel stimulus effectively has an large cost when the correct response is unknown: *analyze is dual to synthesize*. In the case of RPM, this dual route derives the one-to-one correspondence assumption, which often accompanies cognitive models. One route involves working with the algebra's operations (i.e. relations between elements); the other route forgets the operations, which saves time in having to recompute results. We could say that *relation is dual to association*. System and environment are considered broadly to include (pairs of) subsystems within a larger system (e.g., attention and memory within a cognitive system). From the standpoint of expertise, one can see the free-forgetful adjunction as exploiting both domain-relevant relations and a reservoir of learned associations.

One might wonder why we need adjunctions, rather than any pair of alternative routes. The claim is that dual-routes are also systematically, as opposed to arbitrarily related. We have argued that underlying every instance of systematicity is a universal construction of some kind [17]. If, as claimed, that dual-routes are systematically related, then adjunctions (which are another kind of universal construction) provide the basis for a *natural* explanation. Category theory affords general principles in the sense that constructions are typically parametrized by some kind of object, e.g., a category. In this sense, adjoints are a general principle: each of the three examples is based on the same construction parametrized by a different pair of (adjoint) functors.

Although adjoints provide a systematic basis for dual routes, there remains the question of assigning a cost/benefit to each route. As the experimental work on the stimulus-response task suggested, choice of route depends on the task at hand and prior learning [16]. One possibility is to incorporate information theoretic principles, such as a Kolmogorov complexity-based approach to universal artificial intelligence [11]. See [22] for a category theory approach to Kolmogorov complexity. In this way, the route selected is the one with the "shortest" program able to produce the requisite response, which makes the collection of routes an order. Ordered sets are categories with arrows as the order relations; universal morphisms pertain to minimum elements. Though probabilistic models were not considered here, categorical approaches to probability also exist (see, e.g., [2,9]). Providing a categorical explanation for route selection, as well as applications to other instances of dual-routes is a topic of further research.

Acknowledgements. I thank the reviewers for helpful comments. This work was supported by a JSPS Grant-in-aid (16KT0025).

References

1. Aizawa, K.: The systematicity arguments. Studies in Mind and Brain. Kluwer Academic, New York (2003)
2. Avery, T.: Codensity and the Giry monad. J. Pure Appl. Algebra **220**(3), 1229–1251 (2016)
3. Baars, B.J.: The Cognitive Revolution in Psychology. Guilford Press, New York (1986)
4. Calvo, P., Symons, J. (eds.): The Architecture of Cognition: Rethinking Fodor and Pylyshyn's Systematicity Challenge. MIT Press, Cambridge (2014)
5. Carpenter, P.A., Just, M.A., Shell, P.: What one intelligence test measures: a theoretical account of the processing in the Raven Progressive Matrices test. Psychol. Rev. **97**(3), 404–431 (1990)
6. Christoff, K., Prabhakaran, V., Dorfman, J., Kroger, K.J., Zhao, Z., Holyoak, K.J., Gabrieli, J.D.E.: Rostral prefrontal cortex involvement in relational processing during reasoning. Neuroimage **14**(5), 1136–1149 (2001)
7. Evans, J.S.B.T., Stanovich, K.E.: Dual-process theories of higher cognition: advancing the debate. Perspect. Psychol. Sci. **8**(3), 223–241 (2013)
8. Fodor, J.A., Pylyshyn, Z.W.: Connectionism and cognitive architecture: a critical analysis. Cognition **28**(1–2), 3–71 (1988)
9. Fong, B.: Causal theories: a categorical perspective on Bayesian networks (2013). Preprint: arxiv:1301.6201
10. Halford, G.S., Wilson, W.H., Andrews, G., Phillips, S.: Categorizing cognition: toward conceptual coherence in the foundations of psychology. MIT Press, Cambridge (2014)
11. Hutter, M.: Universal artificial intelligence: sequential decisions based on algorithmic probability. Texts in Theoretical Computer Science. Springer, Berlin (2005)
12. Kahneman, D.: Thinking, Fast and Slow. Farrar, Straus and Giroux, NY (2011)
13. Lake, B.M., Ullman, T.B., Tenenbaum, J.B., Gershman, S.J.: Building machines that learn and think like people. Behav. Brain Sci. **24**, 1–101 (2016)
14. Mac Lane, S.: Categories for the working mathematician. Graduate Texts in Mathematics, 2nd edn. Springer, New York (1998)
15. McLaughlin, B.P.: Systematicity redux. Synthese **170**, 251–274 (2009)
16. Phillips, S., Takeda, Y., Sugimoto, F.: Why are there failures of systematicity? The empirical costs and benefits of inducing universal constructions. Front. Psychol. **7**, 1310 (2016)
17. Phillips, S., Wilson, W.H.: Categorial compositionality: a category theory explanation for the systematicity of human cognition. PLoS Comput. Biol. **6**(7), e1000858 (2010)
18. Phillips, S., Wilson, W.H., Halford, G.S.: What do transitive inference and class inclusion have in common? Categorical (co)products and cognitive development. PLoS Comput. Biol. **5**(12), e1000599 (2009)
19. Rich, E.: Artificial intelligence. In: Shapiro, S.C. (ed.) Encyclopedia of Artificial Intelligence, vol. 1, pp. 9–16. Wiley, New York (1987)
20. Smolensky, P.: The constituent structure of connectionist mental states: a reply to Fodor and Pylyshyn. South. J. Philos. **26**, 137–161 (1987)
21. Waltz, J.A., Knowlton, B.J., Holyoak, K.J., Boone, K.B., Mishkin, F.S., Santoa, M.D.M., Thomas, C.R., Miller, B.L.: A system for relational reasoning in human prefrontal cortex. Psychol. Sci. **10**(2), 119–125 (1999)
22. Yanofsky, N.S.: Kolmogorov complexity of categories. In: Coecke, B., Ong, L., Panangaden, P. (eds.) Computation, Logic, Games, and Quantum Foundations. The Many Facets of Samson Abramsky. LNCS, vol. 7860, pp. 350–362. Springer, Heidelberg (2013). doi:10.1007/978-3-642-38164-5_25