Formal Foundations of 3D Geometry to Model Robot Manipulators

Reynald Affeldt [1] Cyril Cohen [2]

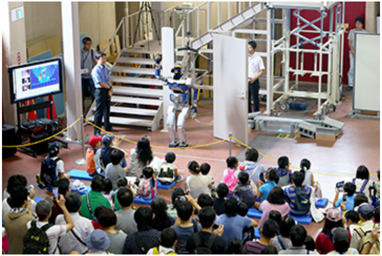
[1]AIST, Japan

[2]INRIA, France

January 16, 2017

Why Verify Robots?

Last summer, I attended a demonstration of the **rescue** capabilities of the HRP-2 robot.



AIST open house in Tsukuba [2016-07-23]

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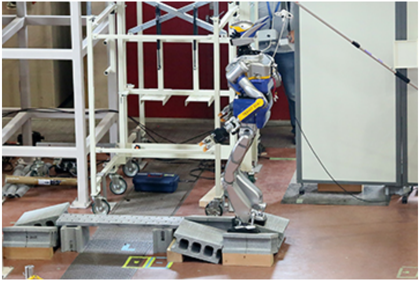
One of the task of the robot was to walk among debris.
In particular, it started walking a very narrow path.



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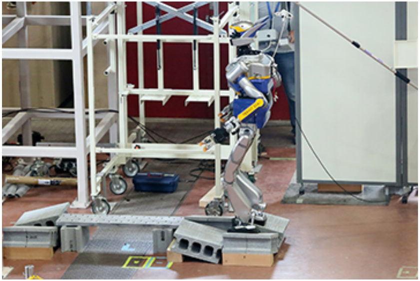

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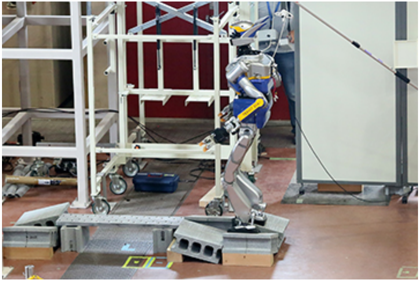

It started walking like this...

. . . **but fell after a few steps**

Motivation and Contribution

- This is a need for safer robots
  - As of today, even a good robot can unexpectedly fail
    - HRP-2 was number 10 among 23 participants at the Finals of the 2015 DARPA Robotics Challenge
- Our work
  - (does not solve any issue with HRP-2 yet)
  - provides formal theories of
    - 3D geometry
    - *rigid body transformations*
  - for describing *robot manipulators*
  - in the Coq proof-assistant [INRIA, 1984∼]

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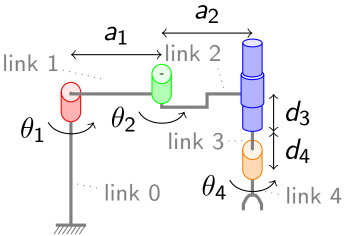

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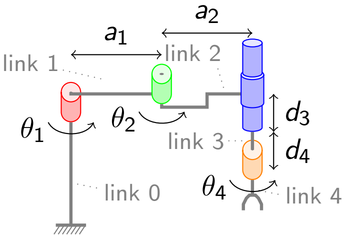

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  - Revolute joint $\leftrightarrow$ rotation
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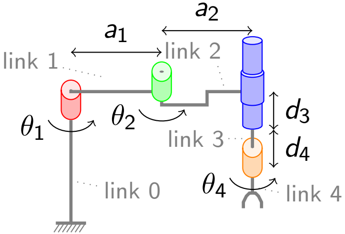

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NB: A humanoid robot can be seen as made of robot manipulators

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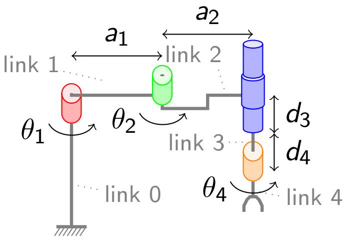

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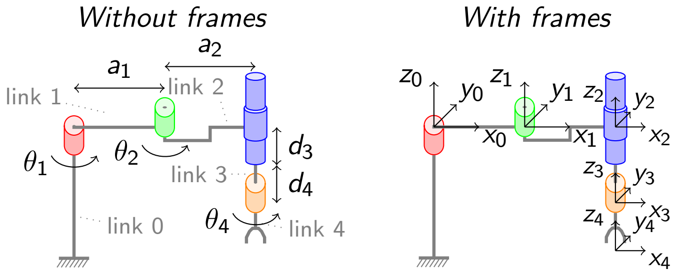

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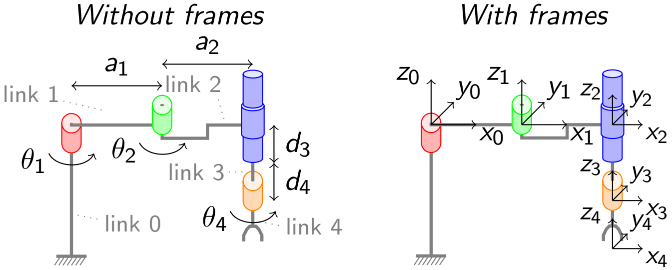


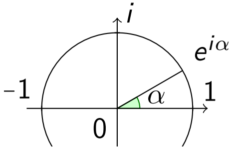
⇒ Approach: use the MATHEMATICAL COMPONENTS library [INRIA/MSR, 2007∼]

- it contains the most extensive formalized theory on matrices and linear algebra

Outline

Formalization of Angles

Basic idea:
angle $\alpha \leftrightarrow$
unit complex number $e^{i\alpha}$



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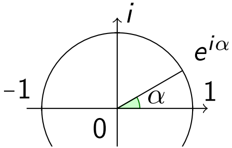

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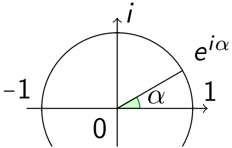

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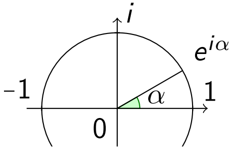

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- Example: definition of π

```
Definition pi := arg (-1).
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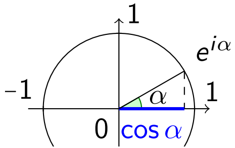

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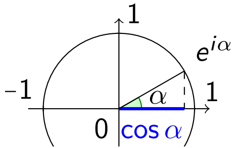

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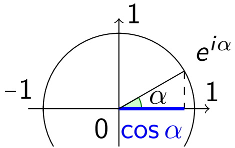

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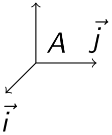
Standard trigonometric relations recovered easily:

- `Lemma acosK x : -1 <= x <= 1 -> cos (acos x) = x.`
- `Lemma sinD a b : sin (a+b) = sin a * cos b + cos a * sin b.`
- ...

Formalization of the Cross-product

The cross-product is used to define oriented frames

$$\vec{k} = \vec{i} \times \vec{j}$$



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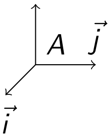

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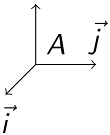

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- Pencil-and-paper definition of the cross-product:

$$\vec{u} \times \vec{v} \stackrel{def}{=} \begin{vmatrix} 1 & 0 & 0 \\ u_0 & u_1 & u_2 \\ v_0 & v_1 & v_2 \end{vmatrix} \text{'e\_0} + \begin{vmatrix} 0 & 1 & 0 \\ u_0 & u_1 & u_2 \\ v_0 & v_1 & v_2 \end{vmatrix} \text{'e\_1} + \begin{vmatrix} 0 & 0 & 1 \\ u_0 & u_1 & u_2 \\ v_0 & v_1 & v_2 \end{vmatrix} \text{'e\_2}$$

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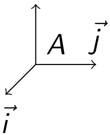

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- Formal definition using Mathematical Components:

```
Definition crossmul u v :=
  \row_(k < 3) \det (col_mx3 'e_k u v).
```

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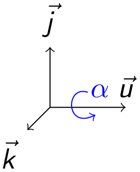
Formal Definition of a Rotation

Rotation of angle α around $\vec{u} \stackrel{def}{=}$

A linear function f and a frame $\langle \frac{\vec{u}}{||\vec{u}||}, \vec{j}, \vec{k} \rangle$ such that:



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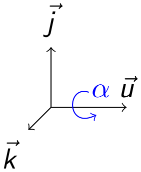

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- Matrices M such that $M\,M^T = 1$ and $\det(M) = 1$
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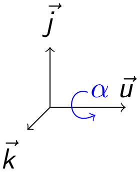

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$\Rightarrow$ Equivalent to rotations defined above

- See the paper for formal proofs

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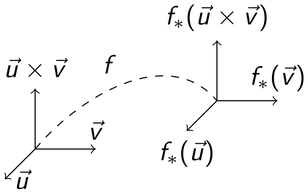

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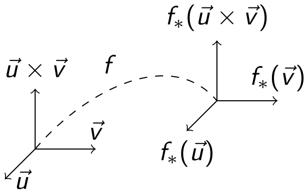

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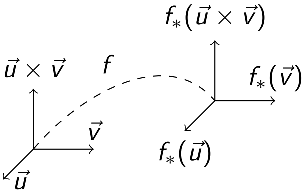

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$\Rightarrow$ Equivalent to *direct isometries*

- See the paper for formal proofs [O'Neill, 1966]

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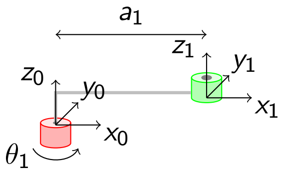

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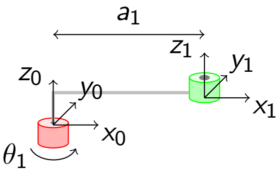

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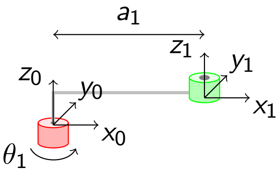

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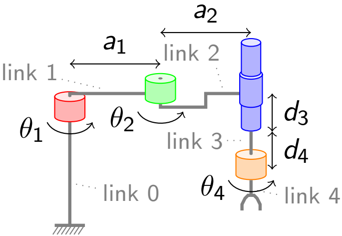
```
Definition A10 :=
  hom (Rz θ1) (row3 (a1 * cos θ1) (a1 * sin θ1) 0).
```

Outline

Forward Kinematics for the SCARA Robot Manipulator

Fwd Kin. = Position and orientation of the end-effector given the link and joint parameters



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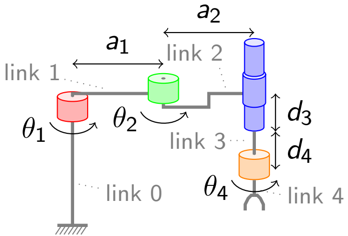

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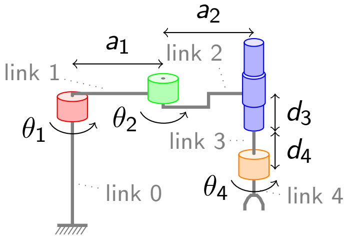

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with

```
Definition scara_rot := Rz (θ1 + θ2+ θ4).
Definition scara_trans := row3
  (a2 * cos (θ2 + θ1) + a1 * cos θ1)
  (a2 * sin (θ2 + θ1) + a1 * sin θ1)
  (d4 + d3).
```

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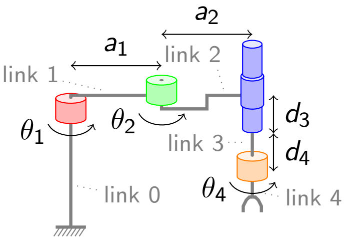
- Example: parameters for the SCARA robot manipulator



link	α_i	a_i	d_i	θ_i
	twist	length	offset	angle
1	0	a_1	0	θ_1
2	0	a_2	0	θ_2
3	0	0	d_3	0
4	0	0	d_4	θ_4

Outline

Exponential Coordinates of Rotations

- Alternative representation with less parameters
- $e^{\alpha S(w)}$ where $S(w) = \begin{bmatrix} 0 & w_z & -w_y \\ -w_z & 0 & w_x \\ w_y & -w_x & 0 \end{bmatrix}$

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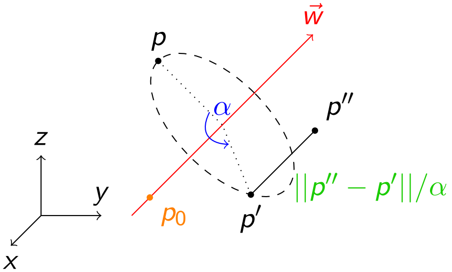
$\Rightarrow$ Equivalent to a rotation of angle α around $\vec{w}$

- See the paper for formal proofs

Outline

What is a Screw Motion?

- An axis (a point and a vector), an angle, a pitch



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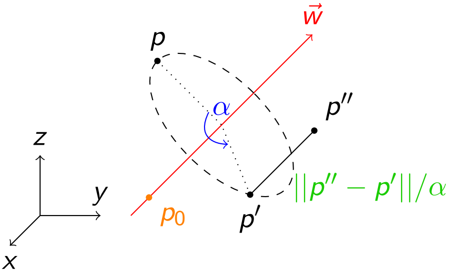

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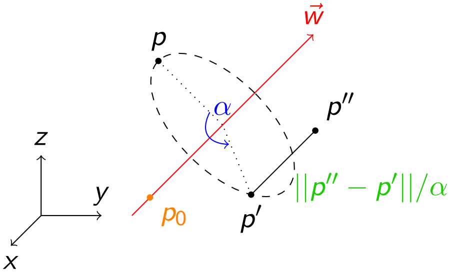

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⇒ Are screw motions RBT?
  - See the paper for Chasles' theorem ("the first theorem of robotics")

Represent Screw Motions with Exponentials of Twists

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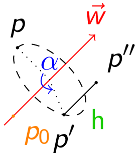

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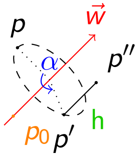

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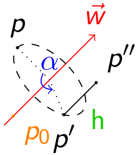
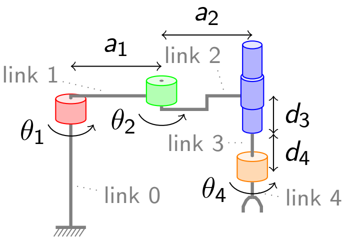


- The pair of vectors (v, w) is called a **twist**
- Luckily, there is a closed formula for $e^{\alpha \begin{bmatrix} S(w) & 0 \\ v & 0 \end{bmatrix}}$

$$\begin{cases} \begin{bmatrix} I & 0 \\ \alpha\, v & 1 \end{bmatrix} & \text{if } w = 0 \\ \begin{bmatrix} e^{\alpha\, S(w)} & 0 \\ \frac{(w \times v)(1 - e^{\alpha\, S(w)}) + (\alpha\, v)(w^T w)}{||w||^2} & 1 \end{bmatrix} & \text{if } w \neq 0 \end{cases}$$

Outline

Fwd Kinematics for SCARA with Screw Motions



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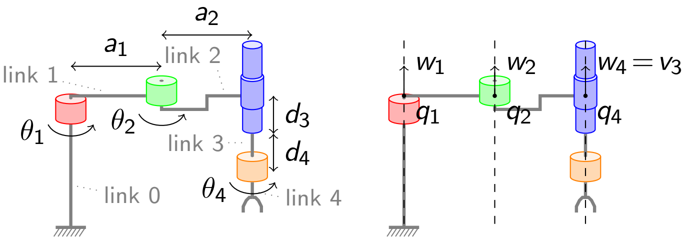

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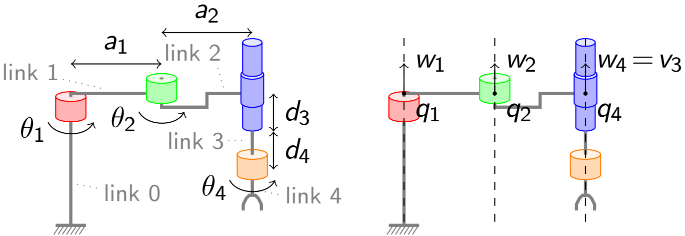

Position and orientation of the end-effector:

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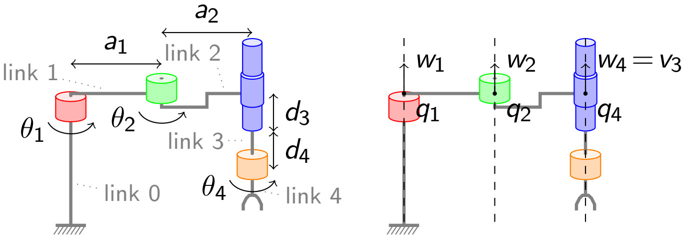

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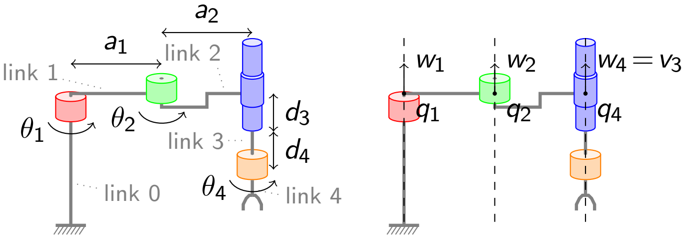

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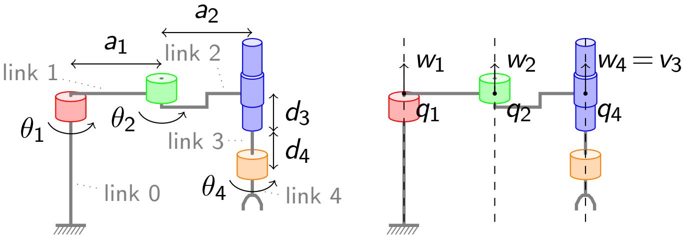

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- Revolute: $t_i = (-w_i \times q_i, w_i)$; prismatic: $t_3 = (v_3, 0)$

Outline

Library Overview

Basic elements of 3D Geometry	
angles	slides 8
trigonometric functions	slide 9
cross-product	slide 10
lines	see the paper
3D Rotations	
using matrices ('SO[R]_3)	slide 12
using exponential coordinates ('so[R]_3)	slide 21
using quaternions	see the paper
using Euler angles	work in progress
Rigid Body Transformations	
using matrices ('SE3[R])	slide 15
using exp. coor. (screw motions) ('se3[R])	slide 24
using dual quaternions	work in progress
using the Denavit-Hartenberg convention	slide 19

⇒ Covers the introductory material of textbooks on robotics

⇒ Enough for forward kinematics of robot manipulators

Related Work

Mostly in 2D

- Collision avoidance algorithm for a vehicle moving in a plane in Isabelle [Walter et al., SAFECOMP 2010]
- Gathering algorithms for autonomous robots and impossibility results
  [Auger et al., SSS 2013] [Courtieu et al., IPL 2015, DISC 2016]
- Event-based programming framework in Coq
  [Anand et al., ITP 2015]
- Planar manipulators in HOL-Light
  [Farooq et al., ICFEM 2013]
- (in 3D) Conformal geometric algebra in HOL-Light
  [Ma et al., Advances in Applied Clifford Algebras 2016]

Future Work

- Various technical improvements
  - Better theory of lines, dependent types to link coordinates with frames
- Instantiate real closed field using classical reals
  - we have been using discrete real closed fields
    - because in MATHEMATICAL COMPONENTS every algebraic structures must have a decidable Leibniz equality
  - yet, equality for classical reals can be assumed decidable
- Application to concrete software
  - by showing preservation of invariants
  - we could use CoRN ideas to bridge with a computable alternative [Kaliszyk and O'Connor, CoRR 2008] [Krebbers and Spitters, LMCS 2011]
  - using CoqEAL for program refinements [Dénès et al., ITP 2012] [Cohen et al., CPP 2013]
- Extension with velocity