Basic results about spaces of sumable sequences in Coq

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For a fixed positive real number p consider the space ℓ_p of sequences whose p-th power is absolutely sumable and on this space consider the p-norm. I.e.

$$\ell_p := \{x\colon \mathbb{N} \to \mathbb{R} \mid \sum |x_n|^p < \infty\}, \quad \text{and} \quad \|x\|_p := \Big(\sum |x_n|^p\Big)^{\frac{1}{p}}.$$

For $(\ell_p, \|\cdot\|_p)$ is a Banach space whenever $1 \leq p \leq \infty$. In particular the Minkowski inequality $\|x+y\|_p \leq \|x\|_p + \|y\|_p$ asserts that the triangle-inequality holds. ℓ^p-spaces are popular as basic examples for Banach-Spaces and extremely well investigated in classical mathematics.

I present formalizations of basic facts about ℓ_p spaces in the proof assistant Coq that I have recently done. In particular I instanciated ℓ_p in Coquelicot's [1] NormedModule type and for this formalized proofs of Minkowski's and Hölder's inequalities and of Young's inequality for products. Just like Coquelicot, the work is based on a classical axiomatization of the real numbers and therefore non-constructive. I did not adhere to Coquelicot's convention to be conservative over this axiomatization and use functional extensionality and proof-irrelevance to avoid the need to replace equality with equivalence relations on spaces of sequences. Some parts of the development also use classical reasoning, for instance those that use the mean value theorem as the current proof from the standard-library is classical. Still, constructive aspects naturally make an appearance as the ℓ_p-norms are best defined as functions into the positive upper reals and the difference between least upper bound and supremum is essential to the definition of the ℓ_p-norms as functions.

The more general L_p-spaces of measurable functions have been formalized in other proof-assistants, for instance in Isabelle/Hol [3]. I used these developments as a guideline for mine. To my knowledge, no treatment in Coq is available. This may be due to additional problems with quotients in Coq, which is also one of the reasons why I decided to restrict to the case ℓ_p where taking the quotient can be done by assuming functional extensionality. I attempted to formulate the results in such a way that they can be reused for more general development of L_p theory at a later point in time.

Some parts of this work may be of separate interest, for instance I developed some basic properties of convex sets and functions. This is because the standard proof of Young's inequality for products relies on the fact that any function on a convex set whose derivative is strictly increasing is already convex. While the result is only needed when applied to the logarithm function, it is proven in full generality in the development. The proof of this statement, in turn, relies on the mean value theorem from the standard library which is restated in a sligthly stronger form that replaces the assumption that the function is continuous in each point of an interval by the weaker assumption that its restriction to the interval is continuous which is closer to the usual mathematical formulation.

The part of this work that is about convex functions can be obtained from a designated github repository [6]. The inequalities for- and definition of ℓ_p is located in the folder for unfinished examples of the Incone library [7]. This is because this work is meant to be the basis of an investigation of ℓ_p spaces from the point of view of computable analysis. Duality theory makes ℓ_p-spaces valuable as examples that can help in developing a toolbox for computability and complexity theory on continuous structures. The Banach-space dual of ℓ_p is ℓ_q, where $1/p + 1/q = 1$ and Banach-Space duality has been investigated in computable analysis [2]. An alternate approach is to consider the dual as a subspace of a function space in the cartesian closed category QCB_0 of quotients of countably based spaces. In this case, the dual of ℓ_p is $\ell_q^{\leq}$ whose topology is more similar to the weak topology and taking the dual again leads back to ℓ_p with the norm topology. The dual pairs $(\ell_p, \ell_q^{\leq})$ consist of a pair of a Banach space and a so-called co-polish space for both of which a complexity theoretical treatment in the sense of computable analysis is available [5, 4]. This is while the construction of the dual space involves taking a function space which is known to be problematic for complexity theoretical treatment [8].

References

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