Analysis of Frictional Forces in Indeterminate Enveloping Grasps

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Abstract—The problem due to the statically indeterminate contact forces arising in indeterminate frictional enveloping grasps is addressed in the paper. First of all, we show that the statical model for the contact forces is incomplete in the sense that the mathematical frictional forces may have an infeasible component. To resolve this infeasibility, we directly derive the enveloping grasp infeasibility inequality which may have an infeasible component. To resolve this infeasibility, we directly derive the enveloping grasp infeasibility inequality which may have an infeasible component. To resolve this infeasibility, we directly derive the enveloping grasp infeasibility inequality which may have an infeasible component. To resolve this infeasibility, we directly derive the enveloping grasp infeasibility inequality which may have an infeasible component.

1 Introduction

Since the introduction of the whole arm manipulation by Salisbury [1], the researchers on manipulation area have been investigating their effort to systematize such a manipulation method and apply it to a variety of systems pertinent to grasp and manipulation [2], [3], [4], [5], [6], [7], [8]. Though the whole-limb grasp/manipulation, or enveloping grasp, may be considered as an alternative to the conventional limb-tip grasp, it provides only a feasible way, depending on the situation, to grasp and manipulate an object. For example, locomotion of a legged robot along a steep terrain [9], grasping multiple objects using a robot hand [10], [11], catching a thrown object using a hand, lifting/capturing a very big and heavy object using a humanoid robot, etc. are such situations.

A major advantage of the enveloping grasp lies in the fact that multiple contacts distributed along multiple limbs work together to generate contact forces which are partly manipulated by control inputs and are partly supported by Nature’s law. Consequently, the enveloping grasp has a tendency of robustness in the sense that an unexpected disturbance acting at the object can be naturally compensated by the change of contact forces which does not require modulation of joint torques [4], [5]. On the other hand, by modulating joint torques, one can impart a certain motion to the object by generating a sufficient wrench [12]. Therefore, it can be said that the enveloping grasp provides a method for robust grasp as well as for manipulable grasp.

Meanwhile, as the number of contact points increases, there arise indeterminate contact forces which are irrelevant to the joint torque or object wrench. The statically indeterminate contact force is the by-product of kinematical over-constrainedness for an enveloping grasp. Firstly, Bicchi [3] characterized the indeterminate forces as preloaded, or wedging, forces for compliant contact model. Later, Omata and Nagata [7] showed that the indeterminate forces for rigid body model are due to frictional forces. Recently, it was shown that the indeterminate contact forces are actually friction forces using a special coordinate transformation for frictional enveloping grasps by Park et al. [13]. This viewpoint which puts a link between indeterminate contact forces and friction forces is very valuable because the physical law governing frictional forces can then be applied to analysis of indeterminate forces. The work by Omata and Nagata would be credited with a first attempt in this line of research, and the work by Park et al. made much advantage of this identification to analyze and synthesize indeterminate enveloping grasps.

This paper addresses the problem due to the statically indeterminate contact forces which occurs inevitably in analysis and synthesis of any indeterminate enveloping grasps. First of all, we show that the frictional forces are actually the by-product of indeterminate forces. Then, the nature of kinematical over-constrainedness in indeterminate grasps is analyzed to give a kinematic constraint on physically feasible sliding velocities. The kinematic constraint on feasible sliding velocities transfers to the kinematic constraint on infeasible frictional forces, which is called the enveloping grasp infeasibility inequality. Then, the recently developed enveloping grasp infeasibility inequality [13] is applied to analyze the region of feasible frictional forces. We give a numerical example which demonstrates how the enveloping grasp infeasibility inequality are applied in cooperation with the predicted feasibility.

2 Physics of Frictional Enveloping Grasps

The figure shown in Fig. 1 illustrates one spatial enveloping grasp. The number of limbs is denoted by \( K \), and each limb is indexed by \( k = 0, 1, 2, \cdots, K \), where \( k = 0 \) implies...
the base limb. We denote the degrees-of-freedom for the
k-th limb by \( n_k \) and the number of contacts distributed
along the k-th limb by \( m_k \). Then the total degrees-of-
freedom is \( N = \sum_{k=0}^{K} n_k \), as \( n_0 = 0 \), and the number of
total contacts is \( M = \sum_{k=0}^{K} m_k \). We reserve the index \( k \)
\((k = 0, 1, \ldots, K)\) for limb indexing. In addition, the index
\( i \) \((i = 1, 2, \ldots, m_k)\) is reserved for contact point indexing,
and the index \( j \) \((j = 1, 2, \ldots, n_k)\) for joint indexing. The
i-th contact of the k-th limb is denoted by \( C_{ki} \). The j-th
joint angle and torque of the k-th limb are denoted by \( q_{kj} \)
and \( \tau_{kj} \), respectively.

Let us attach cartesian coordinates \{ref\} and \{obj\} at
the fixed reference and any convenient point of the object.
The object position, attitude, twist, and wrench are denoted
by the vector \( r_{obj} \in \mathbb{R}^3 \), \( R_{obj} \in SO(3) \), \( \rho \in \mathbb{R}^6 \),
and the vector \( \Omega \in \mathbb{R}^6 \), respectively, all represented with re-
spect to the reference coordinate system \{ref\}. We assume
the point contact with friction for every contact model.
The contact position and force at the contact \( C_{ki} \) are
denoted by \( \chi_{ki} \in \mathbb{R}^3 \) and \( f_{ki} \in \mathbb{R}^3 \), all in \{ref\}. The
contacted joint which is the joint driving the link in contact
with the object is indexed using \( \alpha_k(i) \). The contacted joint
index \( \alpha_k(i) \) is the index of the closest proximal joint to
the contact \( C_{ki} \). In general, \( \alpha_k(m_k) = n_k \), and \( \alpha_0(i) = 0 \)
for \( i = 1, 2, \ldots, m_0 \). For example, the whole limb system
shown in Fig. 1 has \( m_1 = 2 \), \( n_1 = 3 \), and \( \alpha_1(1) = 2 \), \( \alpha_1(2) = 3 \).

### 2.1 Kinematic constraint of frictional enveloping grasps

Let us attach cartesian coordinates \{ref\}, \{obj\}, and
\{\( k\alpha_k(i) \)\} at the fixed reference, the object, and the con-
tacted joint \( \alpha_k(i) \), respectively. For each contact point \( C_{ki} \),
the following holds (see Fig. 1.)

\[
\chi_{ki} = r_{k\alpha_k(i)} + R_{k\alpha_k(i)} \ k_{\alpha_k(i)} \chi_{ki} = r_{obj} + R_{obj} \ \text{obj} \chi_{ki}.
\]

First, note that \( r_{k\alpha_k(i)} \) and \( R_{k\alpha_k(i)} \) are a function of
\( \{q_{k1}, \ldots, q_{k\alpha_k(i)}\} \). If the contact point is not fixed relative
to \( \{k\alpha_k(i)\} \), then \( k_{\alpha_k(i)} \chi_{ki} \) is a function of a parameter
denoted by \( p_{ki} \), that is \( k_{\alpha_k(i)} \chi_{ki} = \phi(p_{ki}) \). If the contact point
is not fixed with respect to \{obj\}, then \( \text{obj} \chi_{ki} = \psi(s_{ki}) \).

Consequently, the following kinematic constraint is de-
rived for the contact \( C_{ki} \)

\[
\dot{\chi}_{ki} = J_{ki} q_{ki} + \Gamma_{ki} \dot{p}_{ki} = G_{ki} T \hat{x} - \Sigma_{ki} \dot{s}_{ki}.
\]

Arranging (2) for every contact of the k-th limb yields

\[
\dot{\chi}_{k} = J_{k} q_{k} + \Gamma_{k} \dot{p}_{k} = G_{k} T \hat{x} - \Sigma_{k} \dot{s}_{k}.
\]

The overall equations for the whole limbs including the
base limb can be arranged as

\[
\dot{\chi} = J q + \Gamma \dot{p} = G T \hat{x} - \Sigma \dot{s}.
\]

### 2.2 Statical constraint of frictional enveloping grasps

From the kinematic constraint (4) the statical duality is applied to yields

\[
\tau = J^T f \quad w = G f
\]

\[
\xi = \Gamma^T f \quad \sigma = \Sigma^T f,
\]

where the vectors \( \xi \) and \( \sigma \) define the generalized forces cor-
responding to \( \dot{p} \) and \( \dot{s} \), respectively. As the latter denotes
the sliding velocity, they represent the frictional forces.
Note that the upper equation is the conventional statical
model for enveloping grasps, whereas the lower one gives
the statical constraint of frictional forces.

On the other hand, the kinematic constraint (4) can be rearranged as

\[
[I \quad \Sigma] \begin{bmatrix} \dot{\chi} \\ \dot{s} \end{bmatrix} = [J \quad \Gamma \quad \Sigma] \begin{bmatrix} \dot{q} \\ \dot{\rho} \\ \dot{s} \end{bmatrix} = G^T \dot{x}.
\]

An enveloping grasp is called normal if \( N = M \). For a nor-
mal grasp with a nonsingular coefficient matrix, the kinem-
atic constraint yields the statical decomposition

\[
f = \Pi_{\Gamma} \tau + \Pi_{\xi} \xi + \Pi_{\sigma} \sigma
\]

\[
w = \Omega_{\tau} \tau + \Omega_{\xi} \xi + \Omega_{\sigma} \sigma
\]

where \( \Pi_{\Gamma} \) \( \Pi_{\xi} \) \( \Pi_{\sigma} \) \( \Omega_{\tau} \) \( \Omega_{\xi} \) \( \Omega_{\sigma} \) \( G \)

The wrench decomposition equation is rearranged to yield the balance-out equation

\[
\Delta w = w - \Omega_{\tau} \tau = [\Omega_{\xi} \quad \Omega_{\sigma}] \begin{bmatrix} \xi \\ \sigma \end{bmatrix}.
\]
2.3 statically indeterminate friction forces

An enveloping grasp is called indeterminate if the degrees-of-indeterminacy defined by

\[ I_{id} = \max \{3M - N - 6, 0\} \]

is positive. For an indeterminate grasp, the balance-out equation (9) can not uniquely determine the frictional forces. As proposed in [13], the frictional force is decomposed into

\[ (\xi, \sigma) = [\Omega_{\xi} \Omega_{\sigma}]^{-1} \Delta w + [N_{\xi} N_{\sigma}] (\xi, \sigma) \]

(10)

using a generalized inverse and a null space bases matrix. When the right-hand side of the balance-out equation (9) is rearranged as

\[ \begin{pmatrix} \Omega_{\xi} & \Omega_{\sigma} \end{pmatrix} \begin{pmatrix} \xi \\ \sigma \end{pmatrix} = \begin{pmatrix} \tilde{\Omega}_{\xi} & \tilde{\Omega}_{\sigma} \end{pmatrix} \begin{pmatrix} \xi \\ \sigma \end{pmatrix} \]

such that the matrix \([\tilde{\Omega}_{\xi} \tilde{\Omega}_{\sigma}]\) becomes a nonsingular square one, we have

\[ \begin{pmatrix} \Omega_{\xi} & \Omega_{\sigma} \\ N_{\xi} & N_{\sigma} \end{pmatrix} = \begin{pmatrix} \tilde{\Omega}_{\xi} & \tilde{\Omega}_{\sigma} \\ - \left( \tilde{\Omega}_{\xi} \tilde{\Omega}_{\sigma} \right)^{-1} \tilde{\Omega}_{\xi} \tilde{\Omega}_{\sigma} \end{pmatrix} \]

(11)

(12)

Therefore, the contact forces in (7) can be further decomposed by

\[ f = \Pi_{\xi} \tau + [\Pi_{\xi} \Pi_{\sigma}] \left( \begin{pmatrix} \Omega_{\xi} \\ \Omega_{\sigma} \end{pmatrix} \right)^{-1} \Delta w \]

\[ + [\Pi_{\xi} \Pi_{\sigma}] \begin{pmatrix} N_{\xi} \\ N_{\sigma} \end{pmatrix} (\xi, \sigma). \]

\[ \]

(13)

It should be noted that the matrix \([\Pi_{\xi} \Pi_{\sigma}] \begin{pmatrix} N_{\xi} \\ N_{\sigma} \end{pmatrix}\) has the property that

\[ \text{Im} \left\{ [\Pi_{\xi} \Pi_{\sigma}] \begin{pmatrix} N_{\xi} \\ N_{\sigma} \end{pmatrix} \right\} = \text{Ker} \left\{ \begin{pmatrix} J^T \\ G \end{pmatrix} \right\}. \]

Therefore, the statically indeterminate contact forces are generated by the indeterminate friction forces \(\tilde{\xi}^T, \tilde{\sigma}^T\).

3 Enveloping Grasp Infeasibility Inequality

3.1 Kinematic constraints on feasible sliding or ready-to-slide velocities

The kinematic constraint (4) can be rearranged as

\[ \begin{pmatrix} J & G^T \end{pmatrix} \begin{pmatrix} \dot{q} \\ -\dot{s} \end{pmatrix} = [\Gamma \Sigma] \begin{pmatrix} -\dot{\hat{\rho}} \\ -\dot{s} \end{pmatrix}. \]

\[ \]

(14)

For an indeterminate enveloping grasp, the left-hand side term embeds a \((N + 6)\)-dimensional subspace in a 3\(M\)-dimensional space. In addition, the right-hand side term generates a 2\(M\)-dimensional subspace in the 3\(M\)-dimensional space. Consequently, the following should hold for the consistency of the constraint

\[ [\Gamma \Sigma] \begin{pmatrix} -\dot{\hat{\rho}} \\ -\dot{s} \end{pmatrix} \in \text{Im} \left\{ [J \ G^T] \right\}. \]

As the matrix \([J^T \ G]\) has a nontrivial null space spanned by \(Z\), i.e. \(\text{Ker} \{ [J^T \ G]\} = \text{Im} \{Z\}\), the equation can be equivalently expressed as

\[ 0 = Z^T [\Gamma \Sigma] \begin{pmatrix} -\dot{\hat{\rho}} \\ -\dot{s} \end{pmatrix}. \]

\[ \]

(15)

Then it is easy to see that the sliding velocity \([-\dot{\hat{\rho}}, -\dot{s}\)\] should belong to the null space of \(Z^T [\Gamma \Sigma] \in \mathbb{R}^{I_{id} \times 2M}\). Because the null space matrix \(Z\) is given by \(Z = [\Pi_{\xi} \Pi_{\sigma}] \begin{pmatrix} N_{\xi} \\ N_{\sigma} \end{pmatrix}\), there follows

\[ \text{Ker} \left\{ Z^T [\Gamma \Sigma] \right\} = \text{Ker} \left\{ [N_{\xi}^T N_{\sigma}^T] \begin{pmatrix} \Pi_{\xi}^T \Pi_{\sigma}^T \end{pmatrix} [\Gamma \Sigma] \right\} \]

\[ = \text{Ker} \left\{ [N_{\xi}^T N_{\sigma}^T] \right\}. \]

The matrix \(S \in \mathbb{R}^{2M \times (2M - I_{id})}\) such that \(\text{Im} \{S\} = \text{Ker} \{[N_{\xi}^T N_{\sigma}^T]\}\) is given by

\[ S = \begin{pmatrix} I \\ \left( \tilde{\Omega}_{\xi} \tilde{\Omega}_{\sigma} \right)^{-1} \left( \tilde{\Omega}_{\xi} \tilde{\Omega}_{\sigma} \right) \end{pmatrix}. \]

\[ \]

(16)

Then the feasible sliding, and ready-to-slide, velocities are given by

\[ \begin{pmatrix} -\dot{\hat{\rho}}^{\text{FEAS}} \\ -\dot{s}^{\text{FEAS}} \end{pmatrix} = Sk \]

for \(k \in \mathbb{R}^{2M - I_{id}}\).

On the other hand, the sliding velocity which has the component not belonging to \(\text{Ker} \{[N_{\xi}^T N_{\sigma}^T]\}\), i.e.

\[ \begin{pmatrix} -\dot{\hat{\rho}}^{\text{INF}} \\ -\dot{s}^{\text{INF}} \end{pmatrix} \in \text{Im} \left\{ \begin{pmatrix} N_{\xi} \\ N_{\sigma} \end{pmatrix} \right\} \]

is not feasible. Therefore, the infeasible sliding velocities are parameterized by

\[ \begin{pmatrix} -\dot{\hat{\rho}}^{\text{INF}} \\ -\dot{s}^{\text{INF}} \end{pmatrix} = \begin{pmatrix} N_{\xi} \\ N_{\sigma} \end{pmatrix} z \]

\[ \]

(17)

for \(z \in \mathbb{R}^{I_{id}}\).
3.2 Kinematic constraint on infeasible frictional forces

The frictional forces which are associated with the infeasible sliding velocities are also infeasible. Observing that a frictional force due to a ready-to-slide velocity, which does not slide in actuality, are in opposite direction, whereas the magnitude is arbitrary only if the frictional cone constraint holds for the frictional force and the normal force. Therefore, the infeasible frictional forces are governed only by a point-wise directional constraint.

As each component of the infeasible frictional force is in the opposite direction to the corresponding component of the infeasible sliding velocity, the infeasible frictional forces are characterized by the following condition: if there exists at least one $z \in \mathbb{R}^{I_{id}}$ such that

$$\text{sign} \left\{ \xi_{ki} \right\} = \text{sign} \left\{ \sigma_{ki} \right\} \quad (18)$$

for the sliding velocity by (17). As the parameter vector $z \in \mathbb{R}^{I_{id}}$ is arbitrary, the condition can be rewritten as

$$\text{sign} \left\{ \xi_{ki} \right\} = \text{sign} \left\{ \sigma_{ki} \right\} \text{sign} \left\{ N_{\xi,ki} \right\} \quad (19)$$

for nonzero $N_{\xi,ki}$ or $N_{\xi,ki}$. If $N_{\xi,ki}$ or $N_{\xi,ki}$ becomes zero, that component is feasible in both direction.

As a matter of fact, the infeasibility condition can be given as a linear inequality, as illustrated in Fig. 2. We call the inequality the enveloping grasp infeasibility inequality.

4 Numerical Example

Consider the normal indeterminate planar grasp shown in Fig. 3, where $q_1 = \left(-135^{-\circ}, -45^{-\circ}\right)$, $q_2 = \left(-45^{-\circ}, -90^{-\circ}\right)$, $\rho_1 = \left(\frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}\right)$, $\rho_2 = \left(\frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}\right)$, $\rho_{11}$, $\rho_{12}$, $\rho_{21}$, and $\rho_{22} = 0$. We set the frictional force at $C_{22}$, i.e. $\xi_{22}$ as the indeterminate frictional force.

1 This is the same numerical example given in Omata and Nagata [7]

![Fig. 2. Enveloping grasp infeasibility inequality](image)

![Fig. 3. A normal-indeterminate grasp: $l_{11} = l_{12} = l_{21} = l_{22} = 2$, $a = 2$, $b = 2 + \sqrt{2}$, $c = \sqrt{2}$](image)

![Fig. 4. Feasible and infeasible sliding velocities](image)

4.1 Physically infeasible range of frictional forces

The feasible sliding velocities are generated by

$$\dot{\rho}^{FEAS} = Sk = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} k_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} k_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} k_3$$

and the infeasible sliding velocities by

$$\dot{\rho}^{INF} = N_\xi z = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} z.$$
we have given by (NFC)  

\[ f_{ki} = f_{kit} + f_{kin}, \]

whose component is given using the geometrical coordinate axes by 

\[ f_{kit} = \text{}'\Pi_{\xi k} t_{ki}, \quad f_{kin} = \text{}'\Pi_{\xi n} n_{ki}. \quad (20) \]

\[ f_{kit} = \text{}'\Pi_{\xi k} t_{ki}, \quad (20) \]

\[ f_{kin} = \text{}'\Pi_{\xi n} n_{ki}. \quad (21) \]

The geometrical coordinate vector \( t_{ki} \) and \( n_{ki} \) are given using the statical variables by 

\[ t_{ki} = \xi_{ki} \quad (22) \]

\[ n_{ki} = \tau_{ki}' = \tau_{ki} + \sum_{j=i+1}^{n_{ki}} i\phi_{\tau kj} \tau_{kj} + \sum_{j=i+1}^{m_{ki}} \phi_{\xi k j} \xi_{kj}, \quad (23) \]

where the influence coefficients \( i\phi_{\tau kj} \) and \( i\phi_{\xi k j} \) are the scalar function of configurations. For the detailed discussion, see the paper [13].

Using the above statical/geometrical coordinate transformation, the tangential and normal force are computed by 

\[ f_{11t} = \begin{pmatrix} 0.7071 \\ -0.7071 \end{pmatrix} + \begin{pmatrix} -0.7071 \\ -0.7071 \end{pmatrix} \xi_{22} \]

\[ f_{12t} = \begin{pmatrix} -0.7071 \\ 0.7071 \end{pmatrix} + \begin{pmatrix} 0.7071 \\ -0.7071 \end{pmatrix} \xi_{22}, \]

\[ f_{21t} = \begin{pmatrix} -1.4142 \\ 1.4142 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \xi_{22}, \]

\[ f_{22t} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -0.7071 \\ -0.7071 \end{pmatrix} \xi_{22} \]

\[ f_{11n} = \begin{pmatrix} 2.8284 \\ -2.8284 \end{pmatrix} + \begin{pmatrix} -1.4142 \\ 1.4142 \end{pmatrix} \xi_{22} \]

\[ f_{12n} = \begin{pmatrix} 1.4142 \\ 1.4142 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \xi_{22}, \]

\[ f_{21n} = \begin{pmatrix} -2.1213 \\ -2.1213 \end{pmatrix} + \begin{pmatrix} 1.4142 \\ 1.4142 \end{pmatrix} \xi_{22}, \]

\[ f_{22n} = \begin{pmatrix} -0.7071 \\ 0.7071 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \xi_{22} \]

Therefore, the enveloping grasp feasibility inequality is given by (NFC) 

\[ n_{11} = 4 - 2\xi_{22} > 0 \quad n_{12} = 2 > 0 \]

\[ n_{21} = -3 + 2\xi_{22} < 0 \quad n_{22} = -1 < 0 \]

As the current grasp is planar, the tangential and normal coordinate axes at the contact \( C_{ki} \) are given by \( \text{'}\Pi_{\xi k} \) and \( \text{'}\Pi_{\xi n} \), which are the \( ki \)-th diagonal block of the matrix \( \Pi_{\xi} \) and \( \Pi_{\xi}, \) respectively. Therefore, the contact force at the contact \( C_{ki} \) is decomposed geometrically into the tangential and normal force [13] 

\[ \xi_{22} < 0 \quad \xi_{22} = 0 \quad 0 < \xi_{22} < 1 \quad \xi_{22} = 1 \quad 1 < \xi_{22} < 2 \quad \xi_{22} > 2 \]

Fig. 5. Behavior of frictional forces according to the variation of \( \xi_{22} \)

or

\[ \xi_{11} < 0 \quad \xi_{12} < 0 \quad \xi_{21} < 0 \quad \xi_{22} < 0. \]

As the friction forces under the balance-out equation are given by

\[ \xi = \Omega_{\xi} (w - \Omega_{\tau} \tau) + N_{\xi} \tilde{\xi} \]

\[ \begin{pmatrix} -1 \\ -1 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \xi_{22} = \begin{pmatrix} -1 + \xi_{22} \\ -1 + \xi_{22} \\ -2 + \xi_{22} \end{pmatrix}, \]

we have

\[ \xi_{22} < 0 \text{ or } \xi_{22} > 2 \]

from the infeasibility inequality. The behavior of the direction of each frictional force in accordance with the indeterminate frictional force \( \xi_{22} \) is illustrated in Fig. 5.

4.2 Physically feasible range of friction forces

The contact forces are decomposed statically as

\[ f = \Pi_{\tau} \tau + \Pi_{\xi} \Omega_{\xi} (w - \Omega_{\tau} \tau) + N_{\xi} \xi_{22}. \]
As we have from (NFC) $\xi < 0$, range of the indeterminate frictional force parameter $\xi$ should have the indeterminate frictional forces yielding the enveloping grasp infeasibility inequality. However, the grasp can stabilize the grasp, since there are indeterminate frictional force satisfying the enveloping grasp feasibility inequality in exclusive of the enveloping grasp infeasibility inequality, as shown in the following example.

Unstable grasp For the case $\mu_1 = \mu_2 = \mu_2 = 1$, the enveloping grasp feasibility inequality yields the range of the indeterminate frictional force parameter $\xi_{22}$

$$-1 \leq \xi_{22} \leq 0.1667.$$  

At first glance, it seems that the current grasp is stable, since there are indeterminate frictional force satisfying enveloping grasp feasibility inequality. However, the grasp can not be stable, as the friction forces are all infeasible in light of the enveloping grasp infeasibility inequality. Stable grasp For the case $\mu_2 = 1$, the range satisfying the enveloping grasp feasibility inequality is given by $-1 \leq \xi_{22} \leq 1$. After eliminating the infeasible frictional forces, we have

$$0 \leq \xi_{22} \leq 1$$

which can stabilize the grasp.

5 Concluding Remarks

In this paper, it was shown that the statically indeterminate contact forces which belong to Ker $\{J \ G^T\}$ can be parameterized using a part of frictional forces. In addition, the overall frictional forces can be computed using the balance-out equation. However, there are some physically infeasible frictional forces which can be identified by the kinematic conditions on direction. Whereas the infeasible sliding velocities are constrained by a linear equation, the infeasible frictional forces are constrained by linear inequalities. As our formulation is deeply rooted in the special coordinate transformation for frictional enveloping grasps, the enveloping grasp infeasibility inequality directly predicts the infeasible frictional forces. This constitutes a main advantage over a previous infeasibility condition proposed in [7]. In addition, the infeasible frictional forces are easily characterized by computing only the signs of the elements of the null space matrix $N_\xi N_\sigma$ once the kinematic parameters are determined, which gives another advantage compared to the previous result where the statical parameters such as the joint torque and wrench and various computations are needed a-priori. Note that this infeasibility in frictional forces appears in indeterminate grasp cases, where the kinematical over-constrainedness is effective. An additional condition for feasible frictional forces was given by the enveloping grasp feasibility inequality, which is due to frictional force constraints. It is worth noting that, as the proposed coordinate transformation has a set of nice geometric properties, the tangential and normal force components can be extracted directly from the kinematic constraint itself without using any geometrical inspection. The enveloping grasp infeasibility and feasibility inequality should be checked at the same time in order to analyze or synthesize an indeterminate enveloping grasp, as demonstrated in the numerical example.

References