Introduction to Tree Language Theory

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IX. Monotone tree automata

Summary of the talks

Talk-9 : Monotone tree automata

1. Monotone tree automata and accepted languages
2. First order theory and the satisfiability
3. Decidable Diophantine arithmetic
4. Open questions

Talk-10 : Monotone A-tree automata & further extensions

5. Tree language hierarchy
6. Closure properties and decidability
7. About ETA families : PTA & TAN

Monotone rules in tree automata

tree automaton $(F, Q, Q_{\text{fin}}, \Delta)$

F : signature

Q : finite set of state symbols, such that $F \cap Q = \varnothing$

Q_{fin} : finite set of final states, such that $Q_{\text{fin}} \subseteq Q$

Δ : finite set of transition rules :

$f(\alpha_1, \ldots, \alpha_n) \rightarrow \beta_1$ [regular rule]

$\alpha_1 \rightarrow \beta_1$ [epsilon rule]

$f(\alpha_1, \ldots, \alpha_n) \rightarrow f(\beta_1, \ldots, \beta_n)$ [monotone rule]

with $f \in F_{(n)}$ & $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in Q$

Note

$\mathcal{C}lass(\text{MTA}_F) = \mathcal{C}lass(\text{TA}_F)$ for any F

MTA & TA

For every monotone TA $\mathcal{A}$, there effectively exists a regular TA $\mathcal{B}$ over the same signature equivalent to $\mathcal{A}$, i.e. $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{B})$

Proof

Let $\mathcal{A} = (F, Q, Q_{\mathsf{fin}}, \Delta)$ be a monotone TA. Define $\mathcal{B} = (F, Q, Q_{\mathsf{fin}}, \Delta')$, where

$\Delta' \quad : \quad f(p_1, \ldots, p_n) \to q \qquad$ if $f(p_1, \ldots, p_n) \to^*_{\Delta} q$ for some $f \in F$, $p_1, \ldots, p_n, q \in Q$

Observe that Δ' is finite, because the number of trees whose height is at most 1 is finite. We show $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{B})$. For "$\subseteq$", let $t = f(t_1, \ldots, t_n)$ in $T_{F \cup Q}$ and $t \to^*_{\Delta} q$ for some $q \in Q$. Then $t \to^*_{\Delta} f(p_1, \ldots, p_n) \to^*_{\Delta} q$. Here we suppose that in the derivation $t \to^*_{\Delta} f(p_1, \ldots, p_n)$, there is no transition step at the root position (though it may happen after $f(p_1, \ldots, p_n)$). This implies $t_i \to^*_{\Delta} p_i$ $(1 \leqslant i \leqslant n)$. Let $J \subseteq \{1, \ldots, n\}$ such that for each $j \in J$, $t_j \in Q$. Define $f(u_1, \ldots, u_n)$ such that $u_i = p_i$ if $i \notin J$; $u_i = t_i$ if $i \in J$. Then $t \to^*_{\Delta} f(u_1, \ldots, u_n) \to^*_{\Delta} f(p_1, \ldots, p_n) \to^*_{\Delta} q$. By induction hypothesis, $t_i \to^*_{\Delta'} u_i$ if $i \notin J$; otherwise, $t_i = u_i$. Moreover, by definition of Δ', we have $f(u_1, \ldots, u_n) \to q$ in Δ'. For the reverse "$\supseteq$", suppose $t \to^*_{\Delta'} q$. Since $\mathcal{B}$ is regular, if t is a state, then $t \equiv q$. For induction step, suppose $t = f(t_1, \ldots, t_n)$. Then $t \to^*_{\Delta'} f(p_1, \ldots, p_n) \to_{\Delta'} q$. Here the final step is performed by a single transition step, and $t_i \to^*_{\Delta'} p_i$ $(1 \leqslant i \leqslant n)$. Then, by induction hypothesis, $t_i \to^*_{\Delta} p_i$ $(1 \leqslant i \leqslant n)$. Moreover, by construction, $f(p_1, \ldots, p_n) \to_{\Delta'} q \in \Delta'$ implies $f(p_1, \ldots, p_n) \to^*_{\Delta} q$. $\square$

Monotone $\mathcal{E}$-tree automata (M $\mathcal{E}$-TA)

equational tree automaton $\mathcal{A} = (\, \mathcal{E}, Q, Q_{\text{fin}}, \Delta \,)$, denoted $\mathcal{A}_{\mathcal{E}}$

$\mathcal{E}$: equational theory (F, E)

F : signature

E : finite set of equations

Q : finite set of state symbols

Q_{fin} : finite set of final states

Δ : finite set of transition rules

In particular,

$\mathcal{A}_{\mathcal{E}}$ is monotone AC-tree automaton if $\mathcal{E}$ is AC-theory

CSL'01, LPAR'05 (Ohsaki *et al.*)

$\mathcal{C}lass(\text{M}\,\mathcal{E}\text{-TA}) \supsetneq \mathcal{C}lass(\mathcal{E}\text{-TA})$ for $\mathcal{E}$ = AC, A, ...

Accepted trees

Given an ETA $\mathcal{A}_\mathcal{E}$: $(\mathcal{E}, Q, Q_{\mathsf{fin}}, \Delta)$

$s \longrightarrow_{\mathcal{A}_\mathcal{E}} t$ (move relation) $\rangle\!\rangle$ $\exists\, l \to r \in \Delta,\ C[\,] \in \mathcal{C}_{F \cup Q}$: $s =_\mathcal{E} C[\,l\,]$ & $t =_\mathcal{E} C[\,r\,]$

t accepted by $\mathcal{A}_\mathcal{E}$ $\rangle\!\rangle$ $\exists\, q \in Q_{\mathsf{fin}}$: $t \longrightarrow_{\mathcal{A}_\mathcal{E}} \cdots \longrightarrow_{\mathcal{A}_\mathcal{E}} q$

$\mathcal{E}$-monotone tree language L $\rangle\!\rangle$ $\exists$ monotone ETA $\mathcal{A}_\mathcal{E}$: $L = \mathcal{L}(\mathcal{A}_\mathcal{E})$

$\mathcal{E}(L)$, called $=_\mathcal{E}$-closure of L $\rangle\!\rangle$ $\{\, s \mid \exists t \in L \colon s =_\mathcal{E} t \,\}$

Remark

$\mathcal{E}(\mathcal{L}(\mathcal{A})) \neq \mathcal{L}(\mathcal{A}_\mathcal{E})$ for monotone AC-TA, while it holds for regular AC-TA :

E.g., consider $\mathcal{A}_\mathcal{E}$ with

Δ : $\mathsf{a} \to \mathsf{q}_1$ $\quad \mathsf{b} \to \mathsf{q}_2$ $\quad \mathsf{c} \to \mathsf{q}_3$ $\quad \mathsf{f}(\mathsf{q}_2, \mathsf{q}_3) \to \mathsf{f}(\mathsf{q}_4, \mathsf{q}_3)$ $\quad \mathsf{f}(\mathsf{q}_1, \mathsf{q}_4) \to \mathsf{q}_5$ $\quad \mathsf{f}(\mathsf{q}_5, \mathsf{q}_3) \to \mathsf{q}$

When q is final state and f is AC-symbol, $\mathcal{A}_\mathcal{E}$ accepts $\mathsf{f}(\mathsf{f}(\mathsf{a},\mathsf{b}),\mathsf{c})$. But $\mathcal{A}$ accepts no tree, and thus $\mathcal{L}(\mathcal{A}) = \varnothing = \mathcal{E}(\mathcal{L}(\mathcal{A})) \neq \mathcal{L}(\mathcal{A}_\mathcal{E})$.

$x \times y \geqslant z$

One can define a monotone AC-TA $\mathcal{A}_{\mathcal{E}}$ over the minimal signature $F = \{\ \mathsf{f}, \mathsf{a}, \mathsf{b}, \mathsf{c}\ \}$ such that $t \in \mathcal{L}(\mathcal{A}_{\mathcal{E}})$ iff $|t|_{\mathsf{a}} \times |t|_{\mathsf{b}} \geqslant |t|_{\mathsf{c}}$.

Proof

Define $\mathcal{A} = (\,\mathcal{E}, Q, Q_{\mathsf{fin}}, \Delta_\alpha \cup \Delta_\beta\,)$,

Δ_α : $\mathsf{a} \to \alpha \quad \mathsf{b} \to \alpha$

$\mathsf{f}(\alpha, \alpha) \to \alpha$

Δ_β : $\mathsf{a} \to \mathsf{q_a} \quad \mathsf{b} \to \mathsf{q_b} \quad \mathsf{c} \to \mathsf{q_c}$

$\mathsf{q_b} \to \mathsf{q_1} \quad \mathsf{q_1} \to \mathsf{q_2} \quad \mathsf{q_2} \to \beta$

$\mathsf{f}(\mathsf{q_2}, \mathsf{q_b}) \to \mathsf{q_1} \quad \mathsf{f}(\mathsf{q_a}, \beta) \to \beta$

$\mathsf{f}(\mathsf{q_a}, \mathsf{q_1}) \to \mathsf{f}(\mathsf{q_{a1}}, \mathsf{q_1})$

$\mathsf{f}(\mathsf{q_{a2}}, \mathsf{q_2}) \to \mathsf{f}(\mathsf{q_a}, \mathsf{q_2})$

$\mathsf{f}(\mathsf{q_{a1}}, \mathsf{q_c}) \to \mathsf{q_{a2}}$

Q_{fin} : $\alpha \quad \beta$

$$\mathsf{f}(\ \mathsf{a}^x, \mathsf{b}^y, \mathsf{c}^z\ )$$
$$\downarrow *$$
$$\mathsf{f}(\ \mathsf{q_a}^x, \mathsf{q_1}, \underbrace{\mathsf{q_b}, \mathsf{q_c}^x, \cdots, \mathsf{q_b}, \mathsf{q_c}^x}_{(m-i)\ \times\ \mathsf{q_b}, \mathsf{q_c}^x}, \mathsf{q_b}^{y-m-1}, \mathsf{q_c}^n\ )$$
$$\downarrow *$$
$$\mathsf{f}(\ \mathsf{q_a}^x, \mathsf{q_2}, \mathsf{q_b}^{y-m-1}\ )$$
$$\downarrow *$$
$$\mathsf{f}(\ \mathsf{q_a}^x, \beta\ )$$

Claim 1 $\quad t \to^*_{\mathcal{A}_{\mathcal{E}}} \alpha$ if $|t|_{\mathsf{c}} = 0$; $t \to^*_{\mathcal{A}_{\mathcal{E}}} \beta$ if $|t|_{\mathsf{a}} \times |t|_{\mathsf{b}} \geqslant |t|_{\mathsf{c}} > 0$

Proof (cont'd)

We prove the reverse of the claim 1. In the previous figure, q_a^x stands for the sequence of q_a of the length x. So in the derivation $t \to^*_{\Delta_\beta/\mathcal{E}} f(q_a^x, q_b^y, q_c^z)$, every constant a (resp. b, c) in t is replaced by q_a (q_b, q_c). The remaining derivation in the previous figure is obtained as the composition of the two derivations

(1) $f(q_a^x, q_1, q_b, q_c^x) \to^*_{\Delta_2/\mathcal{E}} f(q_a^x, q_1)$

(2) $f(q_a^x, q_1, q_c^n) \to^*_{\Delta_2/\mathcal{E}} f(q_a^x, q_2)$

That means, first we obtain from $f(q_a^x, q_b^y, q_c^z)$:

$$f(q_a^x, q_b^{y-1}, q_1, q_c^z) = f(q_a^x, q_1, \overbrace{(q_b, q_c^x), \cdots, (q_b, q_c^x)}^{m}, q_b^{y-m-1}, q_c^n),$$

and then, $f(q_a^x, q_b^{y-1}, q_1, q_c^z) \to^*_{\Delta_2/\mathcal{E}} f(q_a^x, q_2, q_b^{y-m-1})$. The above property (1) follows from the fact that by using the two rules $f(q_a, q_1) \to f(q_{a1}, q_1)$ and $f(q_{a1}, q_c) \to q_{a2}$, we can obtain $f(q_a^x, q_1, q_b, q_c^x) \to^*_{\Delta_2/\mathcal{E}} f(q_{a2}^x, q_1, q_b)$. Then, moreover, we apply $q_1 \to q_2$ and $f(q_{a2}, q_2) \to f(q_a, q_2)$, and finally apply $f(q_2, q_b) \to q_1$. It turns out that $f(q_{a2}^x, q_1, q_b) \to^*_{\Delta_2/\mathcal{E}} f(q_a^x, q_2, q_b) \to^*_{\Delta_2/\mathcal{E}} f(q_a^x, q_1)$. The property (2) is obtained as an immediate consequence of the above discussion. The resulting tree $f(q_a^x, q_b^{y-m-1}, q_2)$ is reachable to $f(q_a^x, \beta^{y-m})$ by using $q_2 \to \beta$. Since $m < y$, there exists at least a β in $f(q_a^x, \beta^{y-m})$, and thus $f(q_a^x, \beta^{y-m}) \to^*_{\Delta_2/\mathcal{E}} \beta$. Hence, as we have seen, given a tree t such that $|t|_a \times |t|_b \geqslant |t|_c$, then t is accepted by $\mathcal{A}_\mathcal{E}$.

In the following we show the claim 1 (the reverse of the above statement), that is, $t \in \mathcal{L}(\mathcal{A}_\mathcal{E})$ implies $|t|_a \times |t|_b \geqslant |t|_c$. (Proof cont'd)

Proof (cont'd)

As an easy observation, one can see that if $t \to^*_{\mathcal{A}_\mathcal{E}} \alpha$, then $|t|_{\mathsf{c}} = 0$. So it suffices to show that for every t in $\mathcal{T}_F$, if $t \to^*_{\mathcal{A}_\mathcal{E}} \beta$, then $|t|_{\mathsf{c}} > 0$ and $|t|_{\mathsf{a}} \times |t|_{\mathsf{b}} \geqslant |t|_{\mathsf{c}}$.

Define the following three mappings for trees in $T_{F \cup Q}$:

$$\begin{aligned} f(t) &= (|t|_{\mathsf{a}} + |t|_{\mathsf{qa}} + |t|_{\mathsf{qa1}} + |t|_{\mathsf{qa2}}) \times (|t|_{\mathsf{b}} + |t|_{\mathsf{qb}} + |t|_{\mathsf{q1}} + |t|_{\mathsf{q2}} + |t|_{\beta}) \\ &\quad -(|t|_{\mathsf{c}} + |t|_{\mathsf{qc}} + |t|_{\mathsf{qa2}}) - (|t|_{\mathsf{q2}} + |t|_{\beta}) \times (|t|_{\mathsf{a}} + |t|_{\mathsf{qa}}) \\ g(t) &= |t|_{\mathsf{q1}} + |t|_{\mathsf{q2}} + |t|_{\beta} \\ h(t) &= (|t|_{\mathsf{c}} + |t|_{\mathsf{qc}}) - |t|_{\mathsf{qa1}} \end{aligned}$$

We claim that $t \to^*_{\mathcal{A}_\mathcal{E}} \beta$ implies $f(t) \geqslant 0$, $g(t) \leqslant 1$, $h(t) \geqslant 0$. By assumption, the rules $\mathsf{a} \to \alpha, \mathsf{b} \to \alpha, \mathsf{f}(\alpha, \alpha) \to \alpha$ in Δ_α can be ignored in the following discussion. We show the above loop invariant property, by induction on the number n of transition steps. If $n = 0$, then $t \equiv \beta$, and thus $f(t) = h(t) = 0$ and $g(t) = 1$. For induction step $n > 0$, suppose $t \to_{\mathcal{A}_\mathcal{E}} t' \to^*_{\mathcal{A}_\mathcal{E}} \beta$. The proof proceeds by case analysis on the transition step $t \to_{\mathcal{A}_\mathcal{E}} t'$. At the transition performed by $\mathsf{a} \to \mathsf{q_a}$ (or $\mathsf{b} \to \mathsf{q_b}$, $\mathsf{c} \to \mathsf{q_c}$), obviously the property holds. In case of the transition step by $\mathsf{q_b} \to \mathsf{q_1}$ (or $\mathsf{q_1} \to \mathsf{q_2}$, $\mathsf{q_2} \to \beta$), we have $g(t') = g(t) + 1$. If $g(t) = 1$, it contradicts to $t \to^*_{\mathcal{A}_\mathcal{E}} \beta$, because $g(t) \leqslant g(u)$ for every u in $t \to^*_{\mathcal{A}_\mathcal{E}} u \to^*_{\mathcal{A}_\mathcal{E}} \beta$. Since $|t|_{\mathsf{q1}} + |t|_{\mathsf{q2}} + |t|_{\beta} = 0$ in this case, we have $f(t) = f(t')$ and $h(t) = h(t')$. (Proof cont'd)

Proof (cont'd)

In case of $\mathsf{f}(\mathsf{q_a}, \beta) \to \beta$, $f(t) - f(t') = (|t'|_{\mathsf{b}} + |t'|_{\mathsf{q_b}} + |t'|_{\mathsf{q_1}} + |t'|_{\mathsf{q_2}} + |t'|_\beta) - (|t'|_{\mathsf{q_2}} + |t'|_\beta) = |t'|_{\mathsf{b}} + |t'|_{\mathsf{q_b}} + |t'|_{\mathsf{q_1}} \geqslant 0$, and we have $g(t) = g(t')$ and $h(t) = h(t')$. In case of $\mathsf{f}(\mathsf{q_{a1}}, \mathsf{q_c}) \to \mathsf{q_{a2}}$, $h(t) - h(t') = -1 + 1 = 0$, and $f(t) = f(t')$ and $g(t) = g(t')$. In case of $\mathsf{f}(\mathsf{q_b}, \mathsf{q_2}) \to \mathsf{q_1}$, we have $f(t) - f(t') = (|t'|_{\mathsf{a}} + |t'|_{\mathsf{q_a}} + |t'|_{\mathsf{q_{a1}}} + |t'|_{\mathsf{q_{a2}}}) - (|t'|_{\mathsf{a}} + |t'|_{\mathsf{q_a}}) = |t'|_{\mathsf{q_{a1}}} + |t'|_{\mathsf{q_{a2}}} \geqslant 0$, and $g(t) = g(t')$ and $h(t) = h(t')$. For monotone transition steps (2 cases), we look carefully. If $\mathsf{f}(\mathsf{q_a}, \mathsf{q_1}) \to \mathsf{f}(\mathsf{q_{a1}}, \mathsf{q_1})$ is applied, $f(t) - f(t') = -(|t'|_{\mathsf{q_2}} + |t'|_\beta)$and $g(t) = g(t')$. Since $g(t') \leqslant 1$ (induction hypothesis) and $|t'|_{\mathsf{q_1}} = 1$, we have $|t'|_{\mathsf{q_2}} = |t'|_\beta = 0$, and thus $f(t) - f(t') = 0$. On the other hand, $h(t') = h(t) - 1$. If $h(t) \leqslant 0$, then extra q_{a1}'s in t' remain in the following derivation, that contradicts to $t \to^*_{\mathcal{A}_\mathcal{E}} \beta$. Thus, $h(t) > 0$, and then $h(t') \geqslant 0$. In case of $\mathsf{f}(\mathsf{q_{a2}}, \mathsf{q_2}) \to \mathsf{f}(\mathsf{q_a}, \mathsf{q_2})$, $f(t) - f(t') = -1 - (|t|_{\mathsf{q_2}} + |t|_\beta) \times (-1)$. Similar to the previous case, since $|t'|_{\mathsf{q_2}} = 1$, we have $|t|_\beta = 0$ by induction hypothesis. Thus, $f(t) - f(t') = -1 + 1 = 0$. Besides, $g(t) = g(t')$ and $h(t) = h(t')$. Hence, if $t \in \mathcal{T}_F$ and $t \to^*_{\mathcal{A}_\mathcal{E}} \beta$, then $|t|_{\mathsf{a}} \times |t|_{\mathsf{b}} \geqslant |t|_{\mathsf{c}}$.

Remark

One can show that $(\Delta_1 \cup \Delta_2)/\mathcal{E}$ is terminating. By taking $\to^*_{\mathcal{A}_\mathcal{E}}$ as a well-founded order, we then can show by induction on this order that "if $f(t) \geqslant 0$, $g(t) \leqslant 1$, $h(t) \geqslant 0$, there exists a derivation $t \to^*_{\mathcal{A}_\mathcal{E}} C[\mathsf{q_2}]$ for some $C \in \mathcal{C}_{\{\mathsf{f},\mathsf{q_a}\}}$," provided that $|t|_{\mathsf{b}} + |t|_{\mathsf{q_b}} + |t|_{\mathsf{q_1}} + |t|_{\mathsf{q_2}} \geqslant 1$ and $|t|_\beta = 0$. This also yields the proof of the claim 1.

$x \times y > z$

One can modify $\mathcal{A}_\mathcal{E}$ to $\mathcal{B}_\mathcal{E}$ such that $t \in \mathcal{L}(\mathcal{B}_\mathcal{E})$ iff $|t|_\mathsf{a} \times |t|_\mathsf{b} > |t|_\mathsf{c}$

Proof

Define $\mathcal{B}_\mathcal{E} = (\mathcal{E}, Q \cup \{\, \mathsf{p_a}, \mathsf{p_b}, \mathsf{q_{b1}}, \mathsf{q_{c1}}, \mathsf{q_{c2}}, \mathsf{q_{f1}}, \mathsf{q_{f2}}, \mathsf{q_{f3}} \,\}, \{\, \mathsf{p_f}, \mathsf{q_{f3}} \,\}, \Delta'_1 \cup \Delta_2 \cup \Delta'_2)$, where

$$\Delta'_1 = \{\, \mathsf{a} \to \mathsf{p_a},\ \mathsf{b} \to \mathsf{p_b},\ \mathsf{f}(\mathsf{p_a}, \mathsf{p_b}) \to \mathsf{p_f},\ \mathsf{f}(\mathsf{p_a}, \mathsf{p_f}) \to \mathsf{p_f},\ \mathsf{f}(\mathsf{p_b}, \mathsf{p_f}) \to \mathsf{p_f} \,\}$$

$$\Delta'_2 = \left\{ \begin{array}{l} \mathsf{f}(\mathsf{q_{a1}}, \mathsf{q_c}) \to \mathsf{q_{c1}},\ \mathsf{f}(\mathsf{q_{a1}}, \mathsf{q_{c1}}) \to \mathsf{q_{f1}},\ \mathsf{f}(\mathsf{q_f}, \mathsf{q_{f1}}) \to \mathsf{q_{f2}} \\ \mathsf{f}(\mathsf{q_{a1}}, \mathsf{q_c}) \to \mathsf{q_{c2}},\ \mathsf{f}(\mathsf{q_b}, \mathsf{q_{c2}}) \to \mathsf{q_{b1}},\ \mathsf{f}(\mathsf{q_b}, \mathsf{q_{b1}}) \to \mathsf{q_{b1}},\ \mathsf{f}(\mathsf{q_{b1}}, \mathsf{q_1}) \to \mathsf{q_{f3}} \end{array} \right\}$$

If $|t|_\mathsf{c} = 0$, then t contains a and b iff t is accepted as $t \to^*_{\mathcal{B}_\mathcal{E}} \mathsf{p_f}$. If $|t| > 0$, we claim that $t \to^*_{\mathcal{B}_\mathcal{E}} \mathsf{q_{f2}}$ iff $|t|_\mathsf{a} \times |t|_\mathsf{b} > |t|_\mathsf{c}$. Let $t = \mathsf{f}(\mathsf{a}^x, \mathsf{b}^y, \mathsf{c}^z)$ and $z = x \times m + (n-1)$ such that $0 \leqslant m < y$ and $0 < n \leqslant x$, i.e. $x \times y > z$, then

$$t \to^*_{\mathcal{B}_\mathcal{E}} \mathsf{f}(\,\mathsf{q_a}^x, \mathsf{q_b}^{y-1}, \mathsf{q_1}, \mathsf{q_c}^z) = \mathsf{f}(\,\mathsf{q_a}^x, \mathsf{q_1}, \overbrace{(\mathsf{q_b}, \mathsf{q_c}^x), \cdots, (\mathsf{q_b}, \mathsf{q_c}^x)}^{m}, \mathsf{q_b}^{y-m-1}, \mathsf{q_c}^{n-1}).$$

If $x = 1$, then $x \times y > z$ iff $y > z$. This implies $t \to^*_{\mathcal{B}_\mathcal{E}} \mathsf{f}(\mathsf{q_{a1}}, \mathsf{q_1}, \mathsf{q_b}^{y-z}, \mathsf{q_c})$. Thus, $\mathsf{f}(\mathsf{q_{a1}}, \mathsf{q_1}, \mathsf{q_b}^{y-z}, \mathsf{q_c}) \to^*_{\mathcal{B}_\mathcal{E}} \mathsf{f}(\mathsf{q_{b1}}, \mathsf{q_1}) \to^*_{\mathcal{B}_\mathcal{E}} \mathsf{q_{f3}}$. If $x \geqslant 2$, from the property (1) in the previous proof, we can erase at most $(x \times m)$ states of $\mathsf{q_c}$ and m states of $\mathsf{q_b}$ from $\mathsf{f}(\,\mathsf{q_a}^x, \mathsf{q_b}^{y-1}, \mathsf{q_1}, \mathsf{q_c}^z)$. Moreover, by using the rules $\mathsf{f}(\mathsf{q_{a1}}, \mathsf{q_c}) \to \mathsf{q_{c1}}$ and $\mathsf{f}(\mathsf{q_{a1}}, \mathsf{q_c}) \to \mathsf{q_{f1}}$, we have $\mathsf{f}(\,\mathsf{q_a}^x, \mathsf{q_1}, \mathsf{q_b}^{y-m-1}, \mathsf{q_c}^{n-1}) \to^*_{\mathcal{B}_\mathcal{E}} \mathsf{f}(\,\mathsf{q_a}^{x-2}, \mathsf{q_2}, \mathsf{q_b}^{y-m-1}, \mathsf{q_{f1}}) \to^*_{\mathcal{B}_\mathcal{E}} \mathsf{f}(\mathsf{q_f}, \mathsf{q_{f1}}) \to_{\mathcal{B}_\mathcal{E}} \mathsf{q_{f2}}$. $\square$

Arithmetic constraints

(1) ψ is exponential Diophantine
if ψ is in D.

(2) ψ is Diophantine
if ψ contains no $(*2)$-formula.

(3) ψ is linear (or Presburger formula)
if ψ contains no $(*1)$-,$(*2)$-formula.

(4) ψ is monotone
if negative sub-formula of ψ
contains no $(*1)$-,$(*2)$-formula.

$$
\begin{aligned}
\mathsf{D} &:= \mathsf{A} \\
&\mid \neg(\ \mathsf{D}\ ) \\
&\mid \mathsf{D} \vee \mathsf{D} \\
&\mid \mathsf{D} \wedge \mathsf{D}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{A} &:= \exists x_i\,(\ \mathsf{D}\ ) \\
&\mid \sum_{i \in I} a_i x_i \geqslant b \\
&\mid x_i x_j \geqslant x_k \quad \cdots \ (*1) \\
&\mid x_i^{x_j} \geqslant x_k \quad \cdots \ (*2)
\end{aligned}
$$

Note

linear $\subsetneq$ monotone (exponential) Diophantine $\subsetneq$ (exponential) Diophantine

Satisfiability

A formula ψ is satisfiable if $\exists\,\theta$ (assignment over $\mathbb{N}$ to free variables in ψ) such that a closed formula $\psi\theta$ is true.

Example

$x^3 \geqslant y$: monotone Diophantine & satisfiable

$\because \Leftrightarrow \exists z\,(\,x \times z \geqslant y \;\wedge\; x \times x \geqslant z\,)$

$x^{x^x} \geqslant y$: monotone exponential Diophantine & satisfiable

$\because \Leftrightarrow \exists z\,(\,x^z \geqslant y \wedge x^x \geqslant z\,)$

$x^2 = y$: Diophantine & not monotone & satisfiable

$\because$ every equivalent formula has a negative sub-formula containing non-linear constraint

e.g. $\Leftrightarrow\; (\,x^2 \geqslant y\,) \;\wedge\; \exists z\,(\,\neg(\,x^2 \geqslant z\,) \;\wedge\; z = y+1\,)$

$x \geqslant y \;\wedge\; y \geqslant x+1$: monotone & not satisfiable

Solution set

Let ψ : arithmetic formula with free variables $x_1, \ldots, x_n$

- the solution set $[\![\, \psi \,]\!]_{\mathbb{N}}$ over the domain $\mathbb{N}$ is the set of vectors

$$[\![\, \psi \,]\!]_{\mathbb{N}} = \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \;\middle|\; \exists c_1, \ldots, c_n \in \mathbb{N} : \psi\,[\, x_1 \mapsto c_1, \ldots, x_n \mapsto c_n \,] \right\}$$

- i-th projection pr_i $(1 \leqslant i \leqslant n)$ is a mapping

$$\mathsf{pr}_i(S) = \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_{i-1} \\ c_{i+1} \\ \vdots \\ c_n \end{pmatrix} \;\middle|\; \exists v \in S : v = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \right\}$$

Cf. cylindrification cy_i $(1 \leqslant i \leqslant n+1)$

Note

$[\![\, \exists x_i\, (\psi) \,]\!] \;=\; \mathsf{pr}_i([\![\, \psi \,]\!])$

Diophantine formulas

For the class of polynomials with integer coefficients, the question if $[\![P(x_1,\ldots,x_n)=0]\!]_{\mathbb{N}} \stackrel{?}{=} \varnothing$ is undecidable [1] $*$ Original Hilbert's 10th problem is the question if $[\![P(x_1,\ldots,x_n)=0]\!]_{\mathbb{Z}} = \varnothing$, and the undecidability result can be extended to $\mathbb{N}$ ([2]).

Corollary

Satisfiability of (exponential) Diophantine formulas is undecidable

Proof

Given a polynomial $P(x_1,\ldots,x_n)$ with integer coefficients, $P(x_1,\ldots,x_n)=0$ is a Diophantine formula. In fact, $P(x_1,\ldots,x_n)$ is represented as $\sum_{i\in I} c_i \times m_i(x_1,\ldots,x_n)$ for some constant $c_i \in \mathbb{Z}$ and monomial m_i ($i \in I$). Then $P(x_1,\ldots,x_n)=0$ iff $\sum_{i\in I} y_i = 0 \ \wedge\ \bigwedge_{i\in I}(c_i \times m_i(x_1,\ldots,x_n) = y_i)$. Note that $a=b$ iff $(a \geqslant b) \wedge \neg(a \geqslant b+1)$. Obviously, $[\![P(x_1,\ldots,x_n)=0]\!]_{\mathbb{N}} \neq \varnothing$ iff $P(x_1,\ldots,x_n)=0$ is satisfiable. However, according to the above undecidability result of Hilbert's 10th problem, satisfiability of $P(x_1,\ldots,x_n)=0$ is undecidable. $\square$

[1] Y. Matiyasevich: *Enumerable sets are Diophantine*, Soviet Mathematics 11, 1970

[2] Y. Matiyasevich: *Hilbert's tenth problem*, MIT Press, Cambridge, 1993

Presburger formulas

Satisfiability of Presburger formulas is decidable

Proof

Define our interpretation of formulas ψ_i with a (finite) set V_i of free variables : $[\![\psi_1 \vee \psi_2]\!]_{\mathbb{N}} = [\![\psi_1]\!]_{\mathbb{N}} \cup [\![\psi_2]\!]_{\mathbb{N}}$; $[\![\neg(\psi_1)]\!]_{\mathbb{N}} = ([\![\psi_1]\!]_{\mathbb{N}})^{\mathsf{c}}$; $[\![\exists x_k(\psi_1)]\!]_{\mathbb{N}} = \mathrm{pr}_k([\![\psi_1]\!]_{\mathbb{N}})$. If $V_1 \neq V_2$, cylindrify elements to a solution set at corresponding variable positions. For instance, if ψ_1 has free variables x_1, x_2 and ψ_2 has a free variable x_1, $[\![\psi_1]\!]_{\mathbb{N}} \cup [\![\psi_2]\!]_{\mathbb{N}}$ means $[\![\psi_1]\!]_{\mathbb{N}} \cup \{ (v, c_2) \mid v \in [\![\psi_2]\!]_{\mathbb{N}}, c_2 \in \mathbb{N} \}$. Note that if $R(x_1, \ldots, x_n)$ is a linear polynomial with integer coefficients, then $[\![R(x_1, \ldots, x_n) \geqslant 0]\!]_{\mathbb{N}}$ is a semi-linear subset of $\mathbb{N}^n$. Moreover, the semi-linearity is closed under Boolean operations (page 6, seminar talk 7) and closed under projection and cylindrification (page 21, seminar talk 7). Thus, if ψ is a Presburger formula, then $[\![\psi]\!]_{\mathbb{N}}$ is semi-linear. Since $[\![\psi]\!]_{\mathbb{N}} = \varnothing$ is decidable for the class of semi-linear sets, so is the satisfiability of ψ. □

Corollary [Ginsburg & Spanier]

A subset S of $\mathbb{N}^n$ is semi-linear iff $\exists$ Presburger formula ψ such that $S = [\![\psi]\!]_{\mathbb{N}}$ with n free variables $\quad * \; [\![\psi]\!]_{\mathbb{N}}$ is called a Presburger set

S. Ginsburg, E.H. Spanier: *Semigroups, Presburger formulas, and languages*, Pacific Journal of Mathematics 16, pp.285–296, 1966

Monotone Diophantine formulas

1. If the class of monotone AC-TA is effectively closed under Boolean operations, the emptiness problem is undecidable

   * "Effective closedness" guarantees not only the existence of effective procedure for the desired computation, but the procedure (the proof for closedness of operations) to be constructable

2. If the class of monotone AC-TA is effectively closed under $\cup, \cap$ and the emptiness problem is decidable, then satisfiability of monotone Diophantine formulas is decidable

Proof of 1

One can show that for every formula $P(x_1, \ldots, x_m) = 0$ with integer polynomial $P(x_1, \ldots, x_m)$, one can find an equivalent formula $\exists \vec{x}_n(\psi)$ for some ψ in S_1 :

$$\mathsf{S}_1 \ ::= \ x \times y \geqslant z \ \mid \ \mathsf{C} \ \text{(Presburger formula)} \ \mid \ \mathsf{S}_1 \vee \mathsf{S}_1 \ \mid \ \mathsf{S}_1 \wedge \mathsf{S}_1 \ \mid \ (\mathsf{S}_1)^{\mathsf{c}}$$

According to the previous result (page 7), there effectively exists a monotone AC-TA that represents $x \times y \geqslant z$. Moreover, every Presburger formula can also be represented by monotone AC=TA (even by regular AC-TA). So if the class of monotone AC-TA is closed under Boolean operations, then one can construct a monotone AC-TA $\mathcal{A}_\mathcal{E}$ such that $\Psi(\mathcal{A}_\mathcal{E}) = [\![\Psi]\!] - \vec{0}$. Note that there is no tree whose Parikh image is $\vec{0}$ (the vector whose elements are all zero). (Proof cont'd)

Proof of 1 (cont'd)

We should notice that $\exists\vec{x}_n\,(\psi)$ is satisfiable iff $\mathcal{L}(\mathcal{A}_\mathcal{E}) \neq \varnothing$ or $\vec{0}$ is a solution of ψ. For a given polynomial $P(x_1,\ldots,x_m)$, (1) the question if $\vec{0} \in [\![P(x_1,\ldots,x_m) = 0]\!]_\mathbb{N}$ is decidable, but (2) the question if $[\![P(x_1,\ldots,x_m) = 0]\!]_\mathbb{N} \stackrel{?}{=} \varnothing$ is not decidable. Hence, if the class of monotone AC-TA is closed under all Boolean operations, then the question if $\mathcal{L}(\mathcal{A}_\mathcal{E}) \stackrel{?}{\neq} \varnothing$ is undecidable (as the question if $\exists\vec{x}_n\,(\psi)$ is satisfiable is undecidable due to (2)). □

Proof of 2

Similar to the previous proof. In this case, we observe that for every monotone Diophantine formula, one can find an equivalent formula $\exists\vec{x}_n(\psi)$ for some ψ in S_2 :

$$\mathsf{S}_2 \;::=\; x \times y \geqslant z \;\mid\; \mathsf{C} \;\;\text{(Presburger formula)}\;\mid\; \mathsf{S}_2 \vee \mathsf{S}_2 \;\mid\; \mathsf{S}_2 \wedge \mathsf{S}_2$$

For every non-linear inequality $x \times y \geqslant z$, there effectively exists a monotone AC-TA that represents $[\![x \times y \geqslant z]\!]_\mathbb{N} - \vec{0}$. Moreover, every Presburger formula can also be represented by monotone AC=TA. So if the class of monotone AC-TA is closed under union and intersection, then one can construct a monotone AC-TA $\mathcal{B}_\mathcal{E}$ such that $\Psi(\mathcal{B}_\mathcal{E}) = [\![\,\Psi\,]\!] - \vec{0}$. Thus, $\exists\vec{x}_n(\psi)$ is satisfiable iff $\mathcal{L}(\mathcal{B}_\mathcal{E}) \neq \varnothing$ or $\vec{0}$ is a solution of ψ. Since the question if $\vec{0}$ is a solution of ψ is decidable, the decidability of $\mathcal{L}(\mathcal{B}_\mathcal{E}) = \varnothing$ implies the decidability of the satisfiability of $\exists\vec{x}_n(\psi)$. □

Closure properties ($\cup, \cap$)

The class of monotone AC-TA is effectively closed under union and intersection

Proof for $\cup$

Obvious. Suppose $\mathcal{E} = (F, E)$ is an AC-theory and $\mathcal{A}_1 = (\mathcal{E}, P, P_{\text{fin}}, \Delta_1)$ and $\mathcal{A}_2 = (\mathcal{E}, Q, Q_{\text{fin}}, \Delta_2)$ are monotone AC-TA whose sets P, Q of state symbols are pairwise distinct. Define $\mathcal{B} = (\mathcal{E}, P \cup Q, P_{\text{fin}} \cup Q_{\text{fin}}, \Delta_1 \cup \Delta_2)$, then by construction, $\mathcal{B}$ accepts a tree t in $\mathcal{T}_F$ iff $\mathcal{A}_1$ or $\mathcal{A}_2$ accepts t. $\square$

Proof for $\cap$

Recall the proof for the closure properties of CSG (pages 14-15, seminar talk 2). Let $\mathcal{A}_1 = (\mathcal{E}, P, P_{\text{fin}}, \Delta_1)$ $\mathcal{A}_2 = (\mathcal{E}, Q, Q_{\text{fin}}, \Delta_2)$ be monotone AC-TA with the same AC-theory $\mathcal{E} = (F, E)$, and denote F_{AC} for the set of AC symbols. Define a monotone AC-TA $B = (\mathcal{E}, (P \cup \{\circ\}) \times Q), P_{\text{fin}} \times Q_{\text{fin}}, \Delta \cup \Delta_{\text{AC}})$ where

$$\Delta: \quad f(\langle p_1, q_1 \rangle, \ldots, \langle p_n, q_n \rangle) \to \langle p, q \rangle \quad \text{if } \exists p_1, \ldots, p_n, p \in P, q_1, \ldots, q_n, q \in Q, f \in F - F_{\text{AC}}: \quad f(p_1, \ldots, p_n) \to^*_{\mathcal{A}_1} p \;\; \& \;\; f(q_1, \ldots, q_n) \to^*_{\mathcal{A}_2} q$$

and

(Proof cont'd)

Proof for $\cap$ (cont'd)

$\Delta_{\mathsf{AC}} = \bigcup_{\mathsf{f} \in F_{\mathsf{AC}}} \Delta_{\mathsf{f}}$ such that for every AC symbol $\mathsf{f} \in F_{\mathsf{AC}}$,

$$\Delta_{\mathsf{f}}: \begin{array}{ll} \mathsf{f}(\langle p_1, q_1 \rangle, \langle p_2, q_2 \rangle) \to \mathsf{f}(\langle p, q_1 \rangle, \langle \circ, q_2 \rangle) & \text{if } \exists \mathsf{f}(p_1, p_2) \to p \in \Delta_1,\, q_1, q_2 \in Q \\ \mathsf{f}(\langle p_1, q_1 \rangle, \langle p_2, q_2 \rangle) \to \mathsf{f}(\langle p_3, q_1 \rangle), \langle p_4, q_2 \rangle) & \text{if } \exists \mathsf{f}(p_1, p_2) \to \mathsf{f}(p_3, p_4) \in \Delta_1,\, q_1, q_2 \in Q \\ \mathsf{f}(\langle \circ, q_1 \rangle, \langle p_2, q_2 \rangle) \to \langle p_2, q \rangle & \text{if } \exists \mathsf{f}(q_1, q_2) \to q \in \Delta_2,\, p_2 \in P \\ \mathsf{f}(\langle \circ, q_1 \rangle, \langle p_2, q_2 \rangle) \to \mathsf{f}(\langle \circ, q_3 \rangle, \langle p_2, q_4 \rangle) & \text{if } \exists \mathsf{f}(q_1, q_2) \to \mathsf{f}(q_3, q_4) \in \Delta_2,\, p_2 \in P \\ \langle p_1, q_1 \rangle \to \langle p_2, q_1 \rangle & \text{if } \exists p_1 \to p_2 \in \Delta_1,\, q_1 \in Q \\ \langle p_1, q_1 \rangle \to \langle p_1, q_2 \rangle & \text{if } \exists q_1 \to q_2 \in \Delta_2,\, p_1 \in P \\ \mathsf{f}(\langle p_1, q_1 \rangle, \langle \circ, q_2 \rangle) \to \mathsf{f}(\langle \circ, q_1 \rangle, \langle p_1, q_2 \rangle) & \text{if } \exists p_1 \in P,\, q_1, q_2 \in Q \\ \mathsf{f}(\langle \circ, q_1 \rangle, \langle p_1, q_2 \rangle) \to \mathsf{f}(\langle p_1, q_1 \rangle, \langle \circ, q_2 \rangle) & \text{if } \exists p_1 \in P,\, q_1, q_2 \in Q. \end{array}$$

Then the monotone AC-TA $\mathcal{B}_{\mathcal{E}}$ simulates $\mathcal{A}_{1\mathcal{E}}$ and $\mathcal{A}_{2\mathcal{E}}$ such that for every tree $t \in \mathcal{T}_F$, $t \to^*_{\mathcal{B}_{\mathcal{E}}} \langle p, q \rangle$ iff $t \to^*_{\mathcal{A}_{1\mathcal{E}}} p$ and $t \to^*_{\mathcal{A}_{2\mathcal{E}}} q$. The above construction is based on the proof of [Theorem3, Ohsaki 2001]. Observe that the same proof argument can be applied to the A-case. That means, by using the same components $\Delta \cup \Delta_{\mathsf{AC}}$ and $(P \cup \{\circ\}) \times Q$ but an A-theory $\mathcal{E}$, one can show that the class of monotone A-TA is closed under intersection. $\square$

H. Ohsaki: *Beyond regularity: Equational Tree Automata for Associative and Commutative Theories*, 15th CSL, LNCS 2142, pp.539–553, 2001

Emptiness problem

The emptiness problem for monotone AC-TA is decidable

Proof

Given a monotone AC-TA $\mathcal{A}_{\mathcal{E}} = (\mathcal{E}, P, P_{\mathsf{fin}}, \Delta_1)$ with $\mathcal{E} = (F, E)$, we define an AC-TA $\mathcal{B}_{\mathcal{E}} = (\mathcal{E}, Q, Q_{\mathsf{fin}}, \Delta_2)$ such that, by letting q be a fresh symbol, $Q = Q_{\mathsf{fin}} = \{\mathsf{q}\}$ and $\Delta_2 = \{f(\mathsf{q}, \ldots, \mathsf{q}) \to \mathsf{q} \mid f \in F\}$. Obviously, $\mathcal{B}_{\mathcal{E}}$ accepts all trees over F. Moreover, $t \in \mathcal{L}(\mathcal{B}_{\mathcal{E}})$ iff $\mathsf{q} \to^*_{\Delta_2^{-1}/\mathcal{E}} t$ where $\Delta_2^{-1} = \{\, s \to t \mid t \to s \in \Delta_2 \,\}$. Let $\Delta_3 = \Delta_1 \cup \Delta_2^{-1}$, then one can show that for every $t \in \mathcal{T}_F$ and $p \in P$, $t \to^*_{\Delta_1/\mathcal{E}} p$ iff $\mathsf{q} \to^*_{\Delta_3/\mathcal{E}} t \to^*_{\Delta_3/\mathcal{E}} p$. It suffices to show that (1) $\to_{\Delta_1/\mathcal{E}} \cdot \to_{\Delta_2^{-1}/\mathcal{E}} \;\subseteq\; \to_{\Delta_2^{-1}/\mathcal{E}} \cdot \to_{\Delta_1/\mathcal{E}}$, (2) $\mathsf{q} \to^*_{\Delta_3/\mathcal{E}} t \to^*_{\Delta_3/\mathcal{E}} p$ implies $\mathsf{q} \to^*_{\Delta_1/\mathcal{E}} t \to^*_{\Delta_2^{-1}/\mathcal{E}} p$. One should note that there exists a final state $p \in P_{\mathsf{fin}}$ such that $\mathsf{q} \to^*_{\Delta_3/\mathcal{E}} p$ iff $\mathcal{L}(\mathcal{A}_{\mathcal{E}}) \neq \varnothing$. Since P_{fin} is finite, there are only finitely many test cases of $\mathsf{q} \to^*_{\Delta_3/\mathcal{E}} p$. Because

the reachability problem for AC-GTRS is decidable (Mayer & Rusinowitch 1998), the question if $\mathcal{L}(\mathcal{A}_{\mathcal{E}}) \neq \varnothing$ is decidable. □

Corollary

Satisfiability of monotone Diophantine formulas is decidable

Corollary

The class of monotone AC-TA is not closed under complement

Proof

For leading to the contradiction that there exists an effective procedure to solve $[\![P(x_1,\ldots,x_n)=0]\!]_{\mathbb{N}}$, we take the hypothesis that for the signature $F=\{\mathsf{f},\mathsf{a},\mathsf{b},\mathsf{c}\}$,

$\mathcal{H}$: $\{\, t \mid |t|_{\mathsf{a}} \times |t|_{\mathsf{b}} \leqslant |t|_{\mathsf{c}} \,\}$ is an AC-monotone tree language.

Given a polynomial $P(x_1,\ldots,x_n)$ with integer coefficients, $P(x_1,\ldots,x_n)=0$ is represented as the conjunction of polynomial equations in the forms of $x_i=a$ ($a\in\mathbb{N}$), $x_i+x_j=x_k$, $x_i\times x_j=x_k$ ($k\neq i,j$). Observe that the equivalent polynomial equations in the above forms may require additional variables $y_1,\ldots,y_m$ ($m\geqslant 0$). Let $F=\{\mathsf{f}\}\cup\{\mathsf{a}_1,\ldots,\mathsf{a}_n,\mathsf{b}_1,\ldots,\mathsf{b}_m\}$. Since the class of AC-monotone tree languages are closed under intersection, an effective procedure can be defined that takes $P(x_1,\ldots,x_n)=0$ as the input and returns a monotone AC-TA $\mathcal{A}_{P\mathcal{E}}$ over F such that $P(x_1,\ldots,x_n)=0$ is satisfiable iff $\mathcal{L}(\mathcal{A}_{P\mathcal{E}})\neq\varnothing$. However, from the undecidability of Hilbert's 10th problem, such a procedure does not exist, and thus $\mathcal{H}$ does not hold. Therefore, the complement of $\{\, t \mid |t|_{\mathsf{a}} \times |t|_{\mathsf{b}} > |t|_{\mathsf{c}} \,\}$ is not AC-monotone. □

Note

C(ETA$_F$) $\subsetneq$ C(META$_F$)

Inclusion problem

The inclusion problem for the class of monotone AC-TA is undecidable

Proof

According to the previous proof, given a polynomial equation $P(x_1,\dots,x_n) = 0$, it can be represented as the conjunction of polynomial equations in the forms of $x_i = a$ $(a \in \mathbb{N})$, $x_i + x_j = x_k$, $x_i \times x_j = x_k$ $(k \neq i, j)$, then suppose $P(x_1,\dots,x_n) = \bigwedge_{1\leqslant i\leqslant \ell} p_i(x_1,\dots,x_n,y_1,\dots,y_m) = q_i$ such that $q_i \in \{x_1,\dots,x_n,y_1,\dots,y_m\}$ or $q_i \in \mathbb{N}$. Let

$$P_{\geqslant}(\vec{x}_n,\vec{y}_m) \triangleq \bigwedge_{1\leqslant i\leqslant \ell} p_i(\vec{x}_n,\vec{y}_m) \geqslant q_i \quad \text{where } (\vec{x}_n,\vec{y}_m) \text{ stands for } (x_1,\dots,x_n,y_1,\dots,y_m)$$

$$Q_k(\vec{x}_n,\vec{y}_m) \triangleq \bigwedge_{i\neq k} p_i(\vec{x}_n,\vec{y}_m) \geqslant q_i \ \wedge p_k(\vec{x}_n,\vec{y}_m) > q_k.$$

Then $P(x_1,\dots,x_n)$ is satisfiable iff $P_{\geqslant}(\vec{x}_n,\vec{y}_m)$ is satisfiable and for all $1 \leqslant i \leqslant \ell$, $Q_i(\vec{x}_n,\vec{y}_m)$ is not satisfiable, because : $[\![P(x_1,\dots,x_n) = 0]\!]_{\mathbb{N}} \neq \varnothing$ iff $[\![P_{\geqslant}(\vec{x}_n,\vec{y}_m)]\!] \not\subseteq \bigcup_{1\leqslant i\leqslant \ell}[\![Q_i(\vec{x}_n,\vec{y}_m)]\!]$. Since $[\![P_{\geqslant}(\vec{x}_n,\vec{y}_m)]\!]$ and $[\![Q_i(\vec{x}_n,\vec{y}_m)]\!]$ $(1 \leqslant i \leqslant \ell)$ are AC-monotone languages and the question if $[\![P(x_1,\dots,x_n) = 0]\!]_{\mathbb{N}} \neq \varnothing$ is undecidable (Hilbert's 10th problem), the inclusion problem for monotone AC-TA is undecidable. □

H. Ohsaki, J.-M. Talbot, S. Tison, Y. Roos: *Monotone AC-tree automata*, 12th LPAR, LNAI 3835, pp.337–351, 2005

Open questions

1. For monotone AC-TA, the question if $\mathcal{L}(\mathcal{A}_{\mathcal{E}}) = \mathcal{T}_F$ is decidable ?
   (universality problem) * open since 2002 (RTA open problem ♯101)

   The universality problem is an instance of the inclusion problem, and the inclusion problem is undecidable. The universality problem is reformulated as $(\mathcal{L}(\mathcal{A}_{\mathcal{E}}))^{\mathsf{c}} \cap \mathcal{T}_F \stackrel{?}{=} \varnothing$, but since the class of AC-monotone tree languages is not closed under complement, $(\mathcal{L}(\mathcal{A}_{\mathcal{E}}))^{\mathsf{c}}$ may not be AC-monotone, though the emptiness problem for monotone AC-TA is decidable and the class of AC-monotone tree languages is closed under intersection.

2. Is there any arithmetic logic that captures monotone AC-TA ?
   (complete logical characterization)

   Recently we showed that a sub-class of monotone (exponential) Diophantine formulas is definable by monotone AC-TA (Theorem 3 in [1]). However, we still do not know exactly which class of the arithmetic can be the counterpart of monotone AC-TA. (Cf. regular AC-TA & Presburger MSO)

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[1] N. Kobayashi & H. Ohsaki: *Tree Automata for Non-Linear Arithmetic*, Proc. of 19th RTA, LNCS 5117, pp.291–305, 2008. (RTA best paper)

Open questions (cont'd)

3. The class of monotone AC-TA is closed under projection ?

   As explained at the beginning, *Petri-nets* are a special class of ground AC-TRS's whose signature is an AC symbol and constants (flat signature). One can easily observe that by definition, every AC-monotone tree language is Petri-net definable. So the interesting question is the reverse : every Petri-net definable tree language is AC-monotone or not? A sufficient condition for the positive answer to this question is that the class of AC-monotone tree languages is closed under projection.

4. One can decide whether a given monotone exponential Diophantine formula is valid ?

   Linearly bounded projection eliminates a certain type of ∃-quantifiers in the formulas, which means that a sub-class of monotone exponential Diophantine formulas is monotone AC-TA definable. In the paper we remain the question whether every monotone exponential Diophantine formula can be transformed to a monotone AC-TA definable formula.

Exercise

1. Show that why the class of monotone AC-TA properly subsumes the class of regular AC-TA.

2. Construct monotone AC-TA over the signature $\{\mathsf{f}, \mathsf{a}, \mathsf{b}, \mathsf{c}\}$ with AC symbol f, whose Parikh's image is :

   (1) $[\![2x + y = z]\!]_{\mathbb{N}} - \vec{0}$

   (2) $[\![x^2 \geqslant y]\!]_{\mathbb{N}} - \vec{0}$

   (3) $[\![2^x \geqslant y]\!]_{\mathbb{N}} - \vec{0}$

   (∗) An arithmetic formula ψ is monotone AC-TA definable iff there effectively exists monotone AC-TA $\mathcal{A}_{\mathcal{E}}$ such that $\Psi(\mathcal{L}(\mathcal{A}_{\mathcal{E}})) = [\![\psi]\!]_{\mathbb{N}} - \vec{0}$.

3. [Linearly bounded formulas] Show that if an arithmetic formula ψ (not restricted to Diophantine formulas) is monotone AC-TA definable, then so is $\exists x\,(x \leqslant y \wedge \psi)$.

   Refer [1] for Exercise 2(3) and 3.

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[1] N. Kobayashi & H. Ohsaki: *Tree Automata for Non-Linear Arithmetic*, Proc. of 19th RTA, LNCS 5117, pp.291–305, 2008.

Appendix (A) : Monotone Exponential Diophantine Formulas

Formulas in D are monotone exponential Diophantine formulas:

$$
\begin{array}{rcllll}
D & ::= & x_i x_j \geqslant x_k & \mid \ x_i^{x_j} \geqslant x_k & \mid \ P & \\
  & \mid & \exists x_i\,(\,\mathsf{D}\,) & \mid \ D \vee D & \mid \ D \wedge D &
\end{array}
$$

$$
\begin{array}{rcllll}
P & ::= & x_i + x_j = x_k & \mid \ x_i = c \ (c \in \mathbb{N}) & & \\
  & \mid & \exists x_i\,(\,\mathsf{P}\,) & \mid \ \forall x_i\,(\,\mathsf{P}\,) & \mid \ P \vee P & \mid \ P \wedge P
\end{array}
$$

The class of formulas $\exists \vec{x}_n\,(\,\psi\,)$ where $\psi \in D'$ coincides with D:

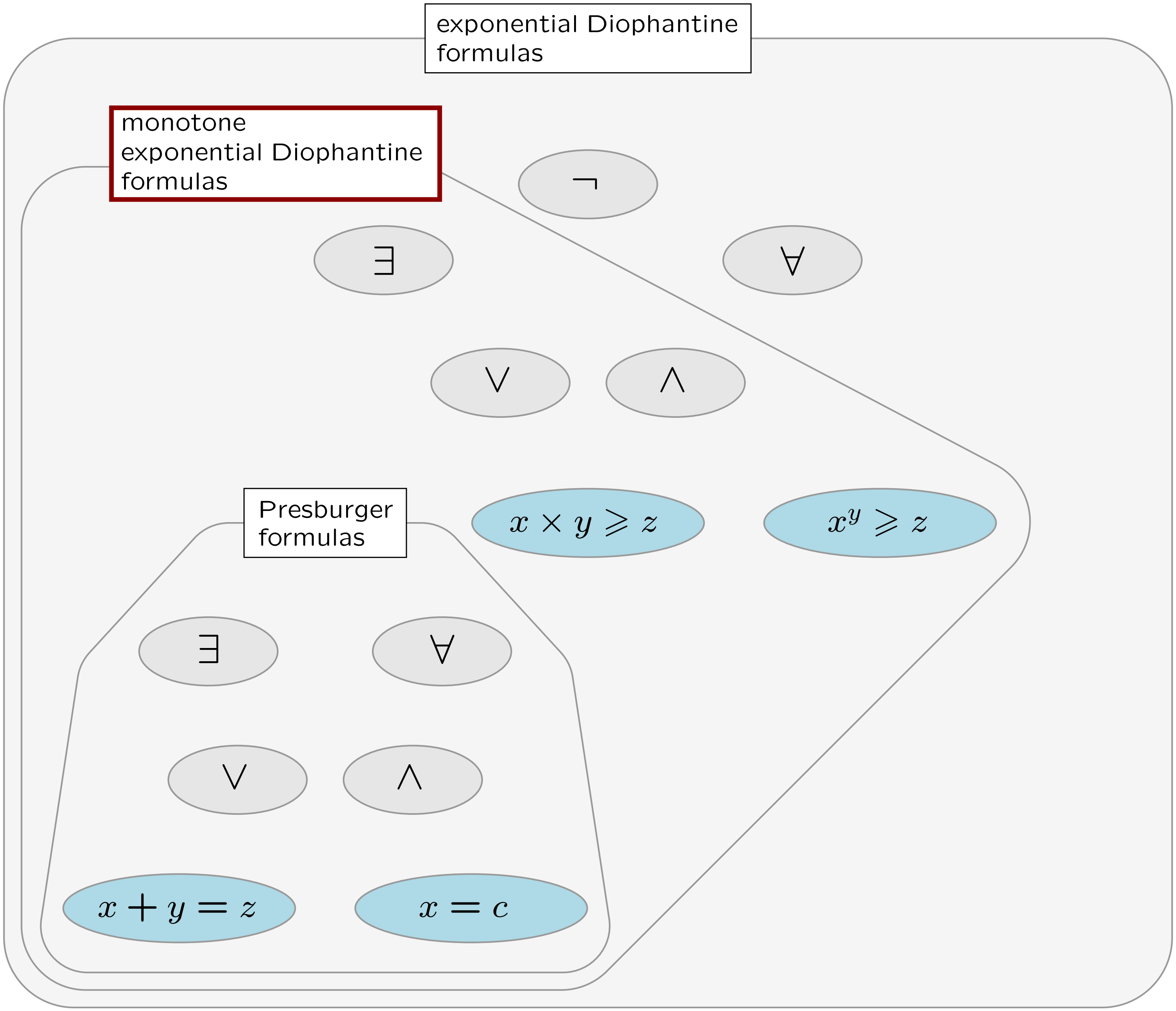
$$
D' \ ::= \ E(\vec{x}_n) \geqslant L(\vec{x}_n) \ \mid \ P \ \mid \ D' \vee D' \ \mid \ D' \wedge D'
$$

where $E(\vec{x}_n)$ is an exponential with non-negative integer coefficients, and $L(\vec{x}_n)$ is a linear polynomial with integer coefficients

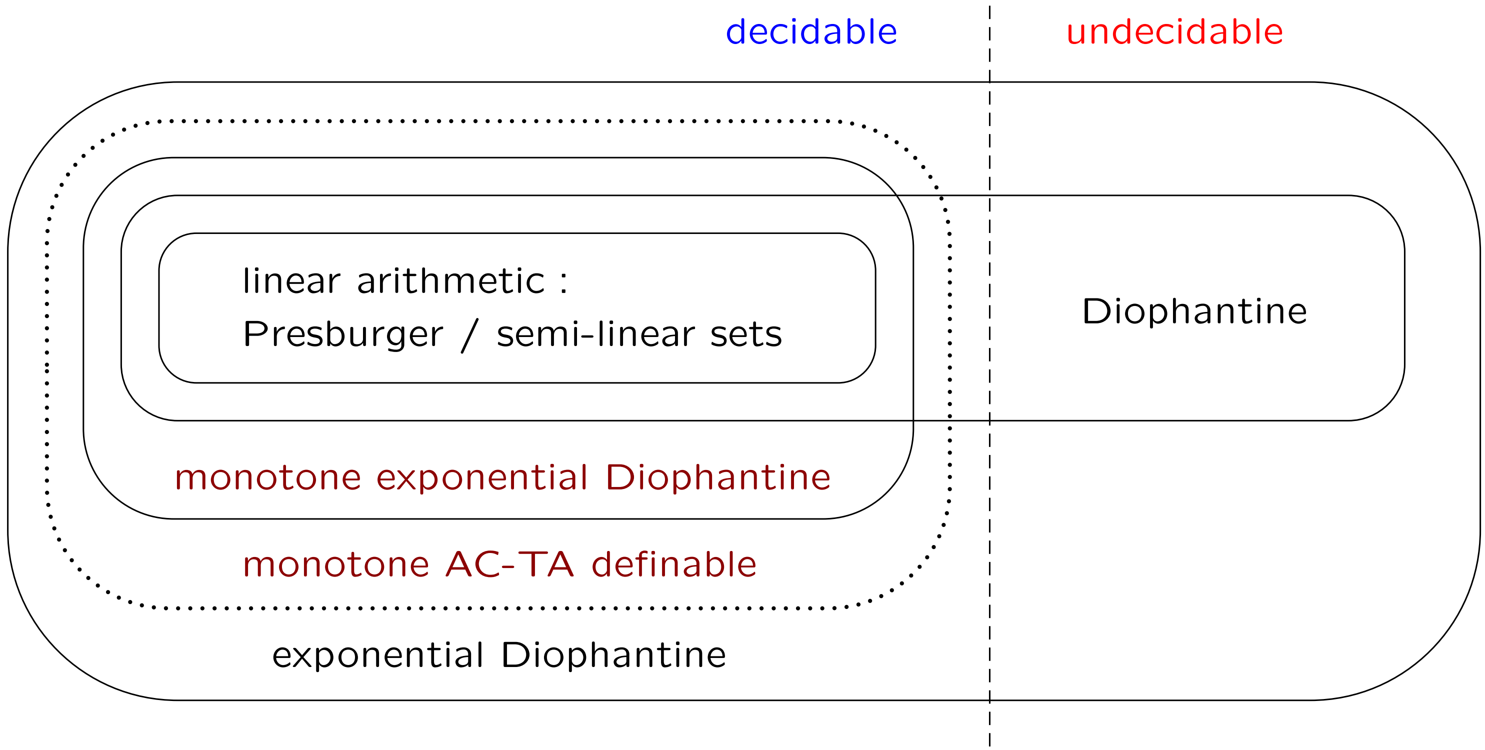
Theorem [Kobayashi & Ohsaki RTA'08]

$E(\vec{x}_n) \geqslant L(\vec{x}_n)$ is definable by monotone AC-TA.

As an immediate consequence, the class D' is definable by monotone AC-TA


exponential Diophantine formulas
monotone exponential Diophantine formulas
¬
∃
∀
∨
∧
Presburger formulas
$x \times y \geqslant z$
$x^y \geqslant z$
∃
∀
∨
∧
$x + y = z$
$x = c$

Appendix (B) : Summary



Solution sets are

- closed under Boolean operations:

  Presburger arithmetic

  (exponential) Diophantine arithmetic

- closed under $\cup$ and $\cap$, but not closed under $(\,)^c$:

  monotone (exponential) Diophantine arithmetic

  monotone AC-TA definable arithmetic (cf. open question 2)

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