Introduction to Tree Language Theory

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VII. Parikh’s theorem
Commutative image

Let \( L \) : language over \( T \) (terminals)

\[ c(L) \] is the **commutative image** of \( L \) if \( \forall u \in \Sigma^* \), \( \exists u \in c(L) \) iff 
\[ \exists w \in L : u, w \text{ are equivalent under the axiom } xy \simeq yx \]

Let \( T = \{a_1, \ldots, a_n\} \)

\( \#a_i(u) \) is the number of occurrences of \( a_i \) in a word \( u \)

\( \#_T(u) \) is the vector \((\#a_1(u), \ldots, \#a_n(u))\)

\( \Psi_T(L) \) is \( \{\#_T(u) \mid u \in L\} \), called **Parikh image** of \( L \)

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**Note**

- \( \Psi_T(L) = \Psi_T(c(L)) \)
- \( \Psi_T(L) = \Psi_T(M) \) if and only if \( c(L) = c(M) \)

\[ \therefore \#_T(u) = \#_T(w) \text{ if and only if } u, w \text{ are equivalent under the axiom } xy \simeq yx \]
Remark

$\mathbb{N}^{|T|}$ (vectors) and $T^*/\sim$ (commutative words) are isomorphic:

\[ \Psi_T : \mathbb{N}^{|T|} \rightarrow \text{mul}(T) \]
\[ g(v) = \bigcup_{i=1}^{|T|} \{ a_i, \ldots, a_i \}_{\text{mul}} \]
\[ g(u+v) = g(u) \cup_{\text{mul}} g(v) \]

\[ h(\varepsilon) = \emptyset \]
\[ h(a) = \{ a_i \}_{\text{mul}} \]
\[ h(xy) = h(x) \cup_{\text{mul}} h(y) \]

\[ \text{mul}(A) : \text{set of multisets over } A, \quad \cup_{\text{mul}} : \text{multiset union}, \quad \{ \cdot \}_{\text{mul}} : \text{multiset} \]
Non-negative vector addition systems (NNVAS)

NNVAS $V = (c, \{v_1, \ldots, v_k\})$ on $\mathbb{N}^n$

- $c$ : vector in $\mathbb{N}^n$, called constant
- $v_1, \ldots, v_k$ : vectors in $\mathbb{N}^n$, called periods

* Originally, VAS [1] is equipped with vectors $v_1, \ldots, v_n$ from $\mathbb{Z}^n$ as periods.

predicate $\Phi_V$ of NNVAS $V$:

$$\Phi_V(v) \iff \exists x_1, \ldots, x_k, \in \mathbb{N} : v = c + (x_1 \times v_1) + \cdots + (x_k \times v_k)$$

set $\llbracket V \rrbracket$ generated by NNVAS $V$:

$$\llbracket V \rrbracket = \{ v \in \mathbb{N}^n \mid \Phi_V(v) \}$$

Note

$\Phi_V(v)$ if and only if $v \in \llbracket V \rrbracket$  ($\Phi_V(v) \equiv \text{true}$ is decidable, so $v \in \llbracket V \rrbracket$ is decidable)

Semi-linear sets

$S$ is a **linear set** if $\exists$ NNVAS $V = (c, \{v_1, \ldots, v_k\})$ such that $S = \llbracket V \rrbracket$

\[
S_1: \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]
\[
S_2: \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

**Semi-linear set** is a finite union $S_1 \cup \cdots \cup S_n$ of linear sets

**Note**

- Linear sets are **not** closed under any Boolean operation. (Exercise)
- The class of semi-linear sets properly includes that of linear sets.
Closure properties

Semi-linear sets are closed under Boolean operations

Proof of $\cup$

Obvious by definition of semi-linearity.

Proof of $\cap$

Let $U = (c, \{u_1, \ldots, u_m\})$ and $V = (d, \{v_1, \ldots, v_n\})$. Define

$A = \{(x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathbb{N}^{m+n} | c + \sum_{1 \leq i \leq m} x_i u_i = d + \sum_{1 \leq j \leq n} y_j v_j \}$

$B = \{(x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathbb{N}^{m+n} | \sum_{1 \leq i \leq m} x_i u_i = \sum_{1 \leq j \leq n} y_j v_j \}$.

One can compute the sets $S_A, S_B$ of minimal positive elements of $A$ and $B - 0$, respectively, and $S_A$ and $S_B$ are finite (Appendix (C)-(1)), where 0 is the vector containing only 0. For each $s \in S_A$, define an NNVAS $W_s = (s, S_B)$. We show that $A = \bigcup_{s \in S_A} \llbracket W_s \rrbracket$ in the following.

For “$\supseteq$”, use the induction on (the structure of) $W_s$. The base case is obvious, because $s$ is a minimal element of $A$. For the induction step, suppose $p \in S_B$ and $q \in \llbracket W_s \rrbracket$ where $q = c + \sum_{1 \leq i \leq m} q(i) u_i = d + \sum_{1 \leq j \leq n} q(m + j) v_j$. Since $p$ is a minimal element of $B$, it satisfies $\sum_{1 \leq i \leq m} p(i) u_i = \sum_{1 \leq j \leq n} p(m + j) v_j$. Then, $p + q = c + \sum_{1 \leq i \leq m} (p(i) + q(i)) u_i = d + \sum_{1 \leq j \leq n} (p(m + j) + q(m + j)) v_j$. Hence, $p + q \in A$.

For “$\subseteq$”, suppose $p \in A$. From minimality, $q \leq p$ for some $q \in S_A$. (Proof cont’d)
Proof of $\cap$ (cont’d)

This implies that: \[ \sum_{1 \leq i \leq m} (p(i) - q(i)) u_i = \sum_{1 \leq i \leq m} p(i) u_i - \sum_{1 \leq i \leq m} q(i) u_i = (d - c) + \sum_{1 \leq j \leq n} p(m + j) v_j - ((d - c) + \sum_{1 \leq j \leq n} q(m + j) v_j) = \sum_{1 \leq j \leq n} p(m + j) v_j - \sum_{1 \leq j \leq n} q(m + j) v_j, \]
and thus, $p - q \in B$. Observe that $B = \llbracket (0, S_B) \rrbracket$; “\(\supseteq\)” is obvious, and “\(\subseteq\)” is shown by structural induction. Since $q + (p - q) \in \llbracket W_q \rrbracket$, $p \in \llbracket W_q \rrbracket$.

Let $f$ be the function from $\mathbb{N}^{m+n}$ to $\mathbb{N}^m$ such that $f(p) = \sum_{1 \leq i \leq m} p(i) u_i$. Obviously, $f(p + q) = f(p) + f(q)$, so $f$ is a linear function (page 21). Since semi-linearity is closed under linear mapping, $\{f(w) \mid w \in A\}$ is semi-linear (Appendix (A)-(2)), and thus, $\llbracket U \rrbracket \cap \llbracket V \rrbracket = \{c + f(w) \mid w \in A\}$ is semi-linear. \[
\]

Proof sketch of $(\cdot)^C$

Suppose that $S$ is a finite union of linear sets $T_1, \ldots, T_n$. By de Morgan’s law, $(S)^C = (T_1)^C \cap \ldots \cap (T_n)^C$. So it suffices to show that the complement of a linear set is semi-linear. Let $V = (c, \{v_1, \ldots, v_n\})$ be an NNVAS on $\mathbb{N}^k$, and define $V_0 = (0, \{v_1, \ldots, v_n\})$. Then, (1) $(\llbracket V_0 \rrbracket)^C$ is semi-linear (Appendix (B)), (2) $X = \{x \in \mathbb{N}^k \mid c \not\leq x\}$ is semi-linear, (3) let $Y = \{y \in \mathbb{N}^k \mid c \leq y\}$, then $(\llbracket V \rrbracket)^C = X \cup (Y - \llbracket V \rrbracket)$. Let $f$ be the function of $\mathbb{N}^k$ such that $f(x) = x + c$. Since $f$ is a bijective function of $\mathbb{N}^k$ onto $Y$, $Y - \llbracket V \rrbracket$ is semi-linear if and only if $f^{-1}(Y - \llbracket V \rrbracket)$ is semi-linear. Observe that $f^{-1}(Y - \llbracket V \rrbracket) = f^{-1}(Y) - f^{-1}(\llbracket V \rrbracket) = \mathbb{N}^k - \llbracket V_0 \rrbracket$. According to (1) & (2), $f^{-1}(Y - \llbracket V \rrbracket)$, which is $(\llbracket V_0 \rrbracket)^C$, is semi-linear, and hence, $X \cup (Y - \llbracket V \rrbracket)$ is semi-linear. \[
\]

Parikh’s theorem

Given CFG $\mathcal{G} = (\Sigma, T, N, q_0, \Delta)$,

1. there exist NNVAS’s $V_1, \ldots, V_k$ such that $\psi_T(L(\mathcal{G})) = \bigcup_{1 \leq i \leq k} [V_i]$, and $V_i$ ($1 \leq i \leq k$) is effectively computable from $\mathcal{G}$
   (Parikh image of context-free language is effectively semi-linear)

2. there exists a regular language $L$ such that $c(L(\mathcal{G})) = c(L)$.
   (Commutative images of the classes of CFL and RL coincide)

Proof

First we show (1) : For each $Q \subseteq N$, define $L_Q(\mathcal{G}) = \{ w \in \Sigma^* \mid w \text{ is obtained from a derivation tree in which every non-terminal symbol of } N \text{ appears} \}$. Observe that $L(\mathcal{G}) = \bigcup_{Q \subseteq N} L_Q(\mathcal{G})$, and thus, $\psi_T(L(\mathcal{G})) = \bigcup_{Q \subseteq N} \psi_T(L_Q(\mathcal{G}))$. Define the conditions 
(a) all $q$ in $Q$ occur in the tree,
(b) no $q$ in $Q$ occurs more than $|N|$-times on any path from the root to a leaf.

And then, define the sets $D_Q$, $\bar{D}_Q$ of derivation trees whose root is $q_0$ :

$D_Q = \{ t \mid \text{derivation tree } t \text{ whose leaves are terminals and that satisfies (a) & (b)} \}$
$\bar{D}_Q = \{ t \mid \text{derivation tree } t \text{ whose leaves are terminals and that satisfies (a)} \}$

(Proof cont’d)
Proof (cont’d)

Moreover, define the set $I_Q$ as follows:

$$I_Q = \left\{ t \mid \text{derivation tree } t \text{ whose root is } q \in Q \text{ and whose leaves contain exactly one non-terminal } q \text{ and that satisfies (b)} \right\}$$

Let

$$V_Q = \bigcup_{s \in D_Q} \left[ \{ \#T(\text{leaf}(s)) \}, \{ \#T(\text{leaf}(t)) \mid t \in I_Q \} \right],$$

then we show that for each $Q \subseteq N$, $V_Q = \Psi_T(L_Q(G))$. For “$\subseteq$”, use the induction on vectors in $V_Q$. In the base case, consider some $s \in D_Q$ such that $\#T(\text{leaf}(s))$ is a constant in $V_Q$. By definition, $\text{leaf}(s) \in L_Q(G)$, and thus, $\#T(\text{leaf}(s)) \in \Psi_T(L_Q(G))$. For induction step, suppose $v_1 \in V_Q$ and $v_2 = \#T(\text{leaf}(t))$ for some $t \in I_Q$. By induction hypothesis, $v_1 \in \Psi_T(L_Q(G))$, and thus, there exists a derivation tree $u \in \tilde{D}_Q$ such that $\text{leaf}(u) \in L_Q(G)$ and $\#T(\text{leaf}(u)) = v_1$. If the root of $t$ is labeled by $q \in Q$ (so $\text{leaf}(t) = w_1qw_2$), then $t = C[q]$. Since $u = C'[w']$ for some $w'$ such that the root of $w'$ is labeled by $q$, the tree $C'[C[w']]$ (obtained by inserting $C$ in between $C'$ and $w'$) is a derivation tree in $\tilde{D}_Q$. Since $\#T(\text{leaf}(C'[C[w']])) = \#T(\text{leaf}(C'[w'])) + \#T(\text{leaf}(C)) = v_1 + \#T(\text{leaf}(t)) = v_1 + v_2$, we obtain $v_1 + v_2 \in \Psi_T(L_Q(G))$.

Next, for “$\supseteq$”, use the induction on trees in $\tilde{D}_Q$. In the base case, consider $t \in D_Q$. By definition, $\#T(\text{leaf}(t)) \in V_Q$. For induction step, suppose $t \in \tilde{D}_Q$ and $t \notin D_Q$ such that every tree $u$ in $\tilde{D}_Q$ smaller than $t$ satisfies $\#T(u) \in V_Q$. (Proof cont’d)
Proof (cont’d)

By assumption, \( t \) has a path (from the root to a leaf) that contains a non-terminal \( q \in Q \) occurring more than \( |Q| \)-times. Let \( n = |Q| \). Then, \( t = C[C_1[\cdots C_m[u] \cdots]] \) such that the root of \( C_i \) (\( 1 \leq i \leq m \) & \( n < m \)) and the root of \( u \) are \( q \). See the right figure. Here \( C \) is possibly the empty context. We will obtain a smaller tree from \( t \) by removing a context among \( C_1, \ldots, C_m \). If some of non-terminals in \( Q \) appears only in \( C_i \), \( C_i \) cannot be a candidate, because the resulting tree is not in \( \tilde{D}_Q \). However, since \( |Q - \{q\}| = n - 1 < m \), there is at least a context, say \( C_\ell \) (yellow part), such that \( t' \in \tilde{D}_Q \) where \( t' = C[C_1[\cdots C_{\ell-1}[C_{\ell+1}[\cdots C_m[u] \cdots]] \cdots]] \). By induction hypothesis, leaf(\( t' \)) \( \in V_Q \). If \( C_k \) satisfies the condition (b), leaf(\( t' \)) + leaf(\( C_\ell \)) \( \in V_Q \), because leaf(\( C_\ell \)) \( \in I_Q \). If there is no such context in \( C_1, \ldots, C_m \), find another non-terminal from \( Q \) that occurs more than \( |Q| \)-times in the same root-leaf path, because this path does not satisfy the condition (b). Repeating this process, one can eventually find a context satisfying (b).

For (2), take the regular grammar \( \mathcal{G}_V \) with production rules \( q_0 \to a_1^{(1)} \cdots a_n^{(n)} | a_1^{v_1(1)} \cdots a_n^{v_1(n)} q_0 | \cdots | a_1^{v_k(1)} \cdots a_n^{v_k(n)} q_0 \) for NNVAS \( V = (c, \{v_1, \ldots, v_k\}) \) on \( \mathbb{N}^n \), where \( v_i(j) \) is \( j \)-th element of vector \( v_i \), then \( \Psi_T(\mathcal{L}(\mathcal{G}_V)) = \|V\| \). This implies that for every semi-linear set \( S \), there exists a regular grammar \( \mathcal{G} \) such that \( \Psi_T(\mathcal{L}(\mathcal{G})) = S \). □
Language inequalities

Let \( \Sigma \) : alphabet with \( T = \{a_1, \ldots, a_m\} \) and \( N = \{x_1, \ldots, x_n\} \)

\( L_\Sigma \) is a set of language components over \( \Sigma \):

\[ \varepsilon, a_1 \ldots, a_m, x_1, \ldots, x_n, \bot, uw, u + w \in L_\Sigma \text{ if } u, w \in L_\Sigma \]

\( f(x_1, \ldots, x_n) \leq x_i \) is a language inequation if \( f(x_1, \ldots, x_n) \in L_\Sigma \)

Let \( L_1, \ldots, L_n \) : languages over \( T \)

\( f(x_1, \ldots, x_n) \) : language components over \( \Sigma \)

\([f(x_1, \ldots, x_n)](L_1, \ldots, L_n)\) is value of \( f(x_1, \ldots, x_n) \):

\[
\begin{align*}
[f](L_1, \ldots, L_n) &= \{f\} \\
[\bot](L_1, \ldots, L_n) &= \emptyset \\
[a_i](L_1, \ldots, L_n) &= \{a_i\} \\
x_i(L_1, \ldots, L_n) &= L_i \\
[uw](L_1, \ldots, L_n) &= [u](L_1, \ldots, L_n) \cdot [w](L_1, \ldots, L_n) \\
[u + w](L_1, \ldots, L_n) &= [u](L_1, \ldots, L_n) \cup [w](L_1, \ldots, L_n)
\end{align*}
\]
Solutions of language inequations

Let \( f_i(x_1, \ldots, x_n) \leq x_i \) : language inequations over \( \Sigma \) (1 \( \leq \) i \( \leq \) n)

\( L_1, \ldots, L_n \) : languages over \( T \)

\((L_1, \ldots, L_n) \) is a solution of \( f_i(x_1, \ldots, x_n) \leq x_i \)

if \( L_i \) is a minimal language satisfying \( [f_i(x_1, \ldots, x_n)](L_1, \ldots, L_n) \subseteq L_i \)

Note 1 (Ginsburg & Rice)

\((L_1, \ldots, L_n) \) is a solution of \( f_i(x_1, \ldots, x_n) \leq x_i \) (1 \( \leq \) i \( \leq \) n) iff \( L_i = \mathcal{L}(G_i) \) such that \( G_i = (\Sigma \cup \{\bot\}, P, \{x_1, \ldots, x_n, \bot\}, x_i, \{ x_i \rightarrow f_i(x_1, \ldots, x_n) \mid 1 \leq i \leq n: f_i(x_1, \ldots, x_n) \leq x_i \} \)

Note 2

We say \((L_1, \ldots, L_n) \) is a solution of \( f_i(x_1, \ldots, x_n) \leq x_i \) over a commutative alphabet if \( L_i \) is a minimal language satisfying \( c([f_i(x_1, \ldots, x_n)](L_1, \ldots, L_n)) \subseteq c(L_i) \). In this definition, \( L_i \) in \((L_1, \ldots, L_n) \) is always a regular language (1 \( \leq \) i \( \leq \) n). However, this is not a consequence of Parikh’s theorem. (Proof is explained later)

Commutative Kleene algebra

Commutative Kleene algebra with variables $X$ is $(A, X, \{+, \cdot, *, 1, 0\})$

$A$ : carrier set
$X$ : finite set of variables

such that the following axioms hold for operators:

- [Associativity]
  $$(x + y) + z = x + (y + z)$$

- [Commutativity of $+$]
  $$x + y = y + x$$

- [Commutativity of $\cdot$]
  $$x \cdot y = y \cdot x$$

- [Distributivity]
  $$x \cdot (y + z) = x \cdot y + x \cdot z$$

- [Identity of $+$]
  $$x + 0 = x$$

- [Identity of $\cdot$]
  $$x \cdot 1 = x$$

- [Idempotency]
  $$x + x = x$$

- [Nullpotency]
  $$x \cdot 0 = 0$$

- [Kleene star]
  $$1 + x \cdot x^* = x^*$$

**Corollary**

$$(x + y)^* = x^* \cdot y^* \quad x + y \cdot z \leq z \Rightarrow x \cdot y^* \leq z \quad (x \leq y \Leftrightarrow x + y = y)$$
Differential operator

Let $K[X]$ : commutative Kleene algebra with variables $X$

$D$ is mapping from $K[X]$ to $K[X]$ such that

(called differential operator)

\[
\begin{align*}
D(x + y) &= D(x) + D(y) \\
D(x \cdot y) &= x \cdot D(y) + y \cdot D(x) \\
D(x^*) &= x^* \cdot D(x) \\
D(1) &= D(0) = 0
\end{align*}
\]

$\frac{\partial}{\partial x}$ is differential operator for $x \in X$ such that

\[
\begin{align*}
\frac{\partial x}{\partial x} &= 1 \\
\frac{\partial y}{\partial x} &= 0 \text{ if } y \in X - \{x\} \\
\frac{\partial a}{\partial x} &= 0 \text{ if } a \in A \\
\frac{\partial}{\partial x}(f(e)) &= \frac{\partial f}{\partial x}(e) \cdot \frac{\partial e}{\partial x} \quad (\frac{\partial f}{\partial x} \text{ is denoted by } f'(x))
\end{align*}
\]
Solution of $f_i(x_1, \ldots, x_n) \leq x_i$ in $K[X]$

Let $f(x_1, \ldots, x_n)$ : finite expression in $K[X]$

$e_1, \ldots, e_n$ : finite expressions in $K[\emptyset]$

$(e_1, \ldots, e_n)$ is solution of $f(x_1, \ldots, x_n) \leq x_i$ in $K[X]$

if $e_i$ is a minimal subset of $A$ satisfying $f(e_1, \ldots, e_n) \subseteq e_i$

Proposition (Hopkins & Kozen 1999)

Every $f(x) \leq x$ in $K[[x]]$ has the unique solution $f'(f(0))^* \cdot f(0)$

Proof
First, we observe that for all polynomials $e, g, h, k$ in $K[X]$,

1. $e(x + g) = e(x) + e'(x + g) \cdot g$
2. $e(g) = e(0) + e'(g) \cdot g$ if $x = 0$

$(e \cdot h \leq g \cdot h \Rightarrow k(e) \cdot h \leq k(g) \cdot h$

Each statement can be shown by the induction on the structure of polynomials.

(Proof cont’d) 15
Proof (cont’d)

Let \( e = g \cdot h, g = f(0), h = f'(g)^* \) in the previous (2), then \( e \cdot h = f(0) \cdot f'(g)^* \cdot f'(g)^* = f(0) \cdot f'(g)^* \) and \( g \cdot h = f(0) \cdot f'(g)^* \). Thus, for any polynomial \( k, k(g \cdot h) \cdot h \leq k(g) \cdot h \) in this case. Therefore, one can conclude that \( f'(f(0)) \cdot f(0) \) satisfies \( f(x) \leq x \):

\[
\begin{align*}
 f(f'(f(0))^* \cdot f(0)) &= f(g \cdot h) \\
 &= f(0) + f'(g \cdot h) \cdot g \cdot h \quad \text{by (1)} \\
 &\leq f(0) + f'(g) \cdot g \cdot h \quad \text{by the above observation} \\
 &= g + f'(g) \cdot g \cdot f'(g)^* \\
 &= (1 + f'(g) \cdot f'(g)^*) \cdot g \\
 &= f'(g)^* \cdot g \quad \text{by [Kleene star]} \\
 &= f'(f(0))^* \cdot f(0)
\end{align*}
\]

For the “least” solution, we show that for every polynomial \( k \) in \( K[[x]] \) that satisfies \( f(k) \leq k, f'(f(0))^* \cdot f(0) \leq k \). According to Corollary in page 13, it suffices to show that \( f(0) + f'(f(0)) \cdot k \leq k \): From monotonicity \( f(0) \leq f(k) \) (as \( 0 \leq k \)) and the assumption \( f(k) \leq k \), one can have \( f'(f(0)) \leq f'(k) \). Thus,

\[
\begin{align*}
 f(0) + f'(f(0)) \cdot k &\leq f(0) + f'(k) \cdot k \\
 &\leq f(k) \quad \text{by (1)} \\
 &\leq k \quad \text{by assumption}
\end{align*}
\]

Hence, uniqueness is justified by the fact that \( f'(f(0))^* \cdot f(0) \) is the least solution. □
Corollary (Generalization of Parikh's theorem)

Every system of inequations \( f_i(x_1, \ldots, x_n) \leq x_i \ (1 \leq i \leq n) \) in \( K[X] \) has the unique solution, and is effectively computable from \( f_1, \ldots, f_n \)

Proof

Suppose \( X = \{x, y\} \), and let the system of inequations : \( f(x, y) \leq x \) and \( g(x, y) \leq y \). First, freeze \( x \), meaning that we consider \( K[\{x\}][\{y\}] \) instead of \( K[\{x, y\}] \). According to the previous proposition, one can compute the (least) solution \( h(x) \) of the inequation \( g(x, y) \leq y \). And then, compute the solution of \( f(x, h(x)) \leq x \) in \( K[\{x\}] \). Let \((k, h(k))\) be the solution obtained by this computation. For the claim that \((k, h(k))\) is the least solution, let \((p, q)\) be a solution of the above system. Since the least solution of \( g(p, y) \leq y \) is \( h(p) \) where \( x \) in \( g(x, y) \) is instantiated by \( p \), \( h(p) \leq q \). By monotonicity and the assumption that \((p, q)\) is a solution, \( f(p, h(p)) \leq f(p, q) \leq p \). Because \( k \) is the least solution of \( f(x, h(x)) \), one can conclude that \( k \leq p \). Therefore,

\[
(k, h(k)) \leq (k, h(p)) \quad \text{by monotonicity} \quad (k \leq p) \\
\leq (p, q) \quad \text{by the above observation} \quad (h(p) \leq q)
\]

Iteratively applying the above computation to the system \( S \) of inequations in \( K[X] \), it results in the least solution of \( S \). \( \square \)
Example

Consider the CFG $G_1, G_2$ with the following production rules:

$\Delta_1 : q_0 \rightarrow aq_0b \quad q_0 \rightarrow \varepsilon$

$\Delta_2 : q_0 \rightarrow aq_1 \quad q_0 \rightarrow bq_2 \quad q_0 \rightarrow \varepsilon$

$q_1 \rightarrow aq_1q_1 \quad q_1 \rightarrow bq_0$

$q_2 \rightarrow aq_0 \quad q_2 \rightarrow bq_2q_2$

Then $L(G_1)$ and $L(G_2)$ are solutions for $x$ of the following systems, respectively:

$S_1 : abx + 1 \leq x$

$S_2 : ay + bz + 1 \leq x \quad ay^2 + bx \leq y \quad ax + bz^2 \leq z$

For $S_1$, according to \textbf{Proposition (Hopkins & Kozen 1999)}

\[ f(x) = abx + 1 \]

\[ f'(f(0))^* \cdot f(0) = (ab)^* \cdot 1 = (ab)^* \] that is equivalent to $L(G_1)$ under commutativity.

For $S_2$, let

\[ f(x, y, z) = ay + bz + 1, \quad g(x, y, z) = ay^2 + bx, \quad h(x, y, z) = ax + bz^2. \]

First, freeze $x, y$, meaning that consider $K[[x]][[y]][[z]]$ for $K[[x, y, z]] : Let \ell_h(z) = ax + bz^2$, then

\[ z = \ell'_h(\ell_h(0))^* \cdot \ell_h(0) = (abx)^* \cdot ax. \]

Next, let $\ell_g(y, z) = ay^2 + bx$, and then consider

\[ \ell_g(y, (abx)^* \cdot ax) = ay^2 + bx. \]

Similar to the previous step, we obtain

\[ y = (abx)^* \cdot bx. \]

Finally, consider

\[ f(x, (abx)^* \cdot bx, (abx)^* \cdot ax) = (abx) \cdot (abx)^* + 1 = (abx)^*. \]

Let $p(x) = (abx)^*$, then computing $p'(x)$ and $p'(p(0))^* \cdot p(0)$ is \textbf{Exercise}. 18
1. Show that the commutative image of a context-free language is not context-free.

2. Construct examples showing the claim in page 5 that the class of linear sets is not closed under union, intersection or complement.

3. Show that semi-linearity is closed under projection, meaning that for every projection $f_i$ such that $f_i(v) = (v(1), \ldots, v(i-1), v(i+1), \ldots, v(k))$, if $S$ is a semi-linear subset of $\mathbb{N}^k$, then $f_i(S) = \{f_i(v) \mid v \in S\}$ is semi-linear.

4. Show that semi-linearity is closed under $\times$, meaning that if $S$ and $T$ are semi-linear subsets of $\mathbb{N}^m$ and of $\mathbb{N}^n$, then $S \times T$ is semi-linear.

5. Construct an example showing that for a language $L$ over $T$, $\Psi_T(L)$ is semi-linear but $L$ is not context-free.

6. Show that $\{w \in \{a, b\}^* \mid |w|_a = (|w|_b)^2 \}$ is not context-free.

7. Show that every context-free language over a one-letter alphabet is regular.

8. Compute $p'(x)$ and the solution for $x, y, z$ of $S_2$ in page 18.
Appendix (A): Basic properties of semi-linear sets

(1) Every linear set on $\mathbb{N}^k$ is a finite union of linear sets on $\mathbb{N}^k$, each of which is linearly independent periods.

Proof

Use the induction on the number of periods. The base case is obvious, because a linear set (obtained by an NNVAS) with one period satisfies (1). For induction step, let $L = \llbracket V \rrbracket$ where $V = (c, \{v_1, \ldots, v_n\})$, and suppose $v_1, \ldots, v_n$ is linearly dependent periods.

Then, there exist a permutation $\pi$ over $\{1, \ldots, n\}$, non-negative integers $t_i$ ($1 \leq i \leq k$) and positive integers $t_j$ ($k < j \leq n$) such that $\sum_{1 \leq i \leq k} t_i v_{\pi(i)} = \sum_{k+1 \leq j \leq n} t_j v_{\pi(j)}$ (*). For each $j$ ($k < j \leq n$), define $C_j = \{c + xv_{\pi(j)} | 0 \leq x < t_j\}$ and $P_j = \{v_1, \ldots, v_n\} - \{v_{\pi(j)}\}$.

Let $L_j = \bigcup_{0 \leq x < t_j} \llbracket(c + xv_{\pi(j)}, P_j)\rrbracket$, then by induction hypothesis, $L_j$ is the finite union of linear sets, each of which satisfies (1).

Next, we show that $L = \bigcup_{k < j \leq n} L_j$. By construction, $L_j \subseteq L$ ($k < j \leq n$), and thus, $\bigcup_{k < j \leq n} L_j \subseteq L$. For “$\subseteq$”, let $v = c + \sum_{1 \leq i \leq n} d_i v_{\pi(i)}$ in $\mathbb{N}^k$. If $d_j \geq t_j$ ($k < j \leq n$), then take $u_1 = c + \sum_{1 \leq i \leq k} (d_i + t_i) v_{\pi(i)} + \sum_{k < j \leq n} (d_j - t_j) v_{\pi(j)}$. From (*), $v = u_1$. After $\ell$-times application of the above procedure, one can obtain $u_\ell$ such that $v = u_\ell$ and a coefficient of $v_{\pi(j)}$ is less than $t_j$ for some $j$ ($k < j \leq n$). Let $u_\ell = c + \sum_{1 \leq i \leq n} e_i v_{\pi(i)}$, then $u_\ell \in L_j$, because $c + e_j v_{\pi(j)} \in C_j$ and $\{v_1, \ldots, v_n\} - \{v_{\pi(j)}\} = P_j$.

As a corollary of (1), it follows that: every semi-linear set on $\mathbb{N}^k$ is a finite union of linear sets on $\mathbb{N}^k$, each of which is linearly independent periods.
Appendix (A): Basic properties of semi-linear sets (cont’d)

A function $f$ from $\mathbb{N}^m$ to $\mathbb{N}^n$ is linear if $f(x + y) = f(x) + f(y)$.

(2) Semi-linearity is closed under linear mapping, meaning that for every linear function $f$ from $\mathbb{N}^m$ to $\mathbb{N}^n$, if $S$ is a semi-linear subset of $\mathbb{N}^m$, then $f(S) = \{f(v) \mid v \in S\}$ is semi-linear.

**Proof**

It suffices to show that linearity is closed under linear mapping. Let $L = [V]$ where $V = (c, \{v_1, \ldots, v_k\})$, then $f(L) = [V']$ where $V' = (f(c), \{f(v_1), \ldots, f(v_k)\})$. So, for every $x \in L$, $f(x) \in f(L)$. Conversely, if $u \in f(L)$, then $u = f(c) + \sum_{1 \leq i \leq k} y_if(v_i) = f(c + \sum_{1 \leq i \leq k} y_iv_i)$. Hence, $u = f(z)$ for some $z \in L$. \qed

(3) Semi-linearity is closed under inverse linear-mapping, meaning that if $S$ is a semi-linear subset of $\mathbb{N}^m$, then $f^{-1}(S) = \{v \in \mathbb{N}^m \mid \exists f(v) \in S\}$ is semi-linear.

**Proof**

For $p = (x_1, \ldots, x_a)$ and $q = (y_1, \ldots, y_b)$, we denote $p \times q$ for $(x_1, \ldots, x_a, y_1, \ldots, y_b)$. Let $g$ be the function $g(x) = x \times f(x)$. From linearity of $f$, we have $g(x + y) = (x + y) \times (f(x) + f(y)) = (x \times f(x)) + (y \times f(y)) = g(x) + g(y)$. So, $g$ is a linear function. From (2), $g(\mathbb{N}^m)$ is a semi-linear subset of $\mathbb{N}^{m+n}$. Moreover, since $\mathbb{N}^m \times S$ is semi-linear, $g(\mathbb{N}^m) \cap (\mathbb{N}^m \times S)$ is semi-linear, because semi-linearity is closed under intersection. Let $h$ be the projection $h(x \times y) = x$, then $h(g(\mathbb{N}^m) \cap (\mathbb{N}^m \times S)) = f^{-1}(S)$. Hence, $f^{-1}(S)$ is semi-linear, because semi-linearity is closed under projection. \qed
Appendix (B) : Complement of semi-linear sets

For every NNVAS $V = (0, \{v_1, \ldots, v_n\})$ on $\mathbb{N}^k$ with linearly independent vectors $v_1, \ldots, v_n$, $(\mathbb{N}^k)^c$ is semi-linear. (0 is the vector containing only 0)

Proof

If $k > n$, then find a mapping $\pi$ from $\{1, \ldots, k - n\}$ to $\{1, \ldots, k\}$ such that for unit vectors $e_{\pi(1)}, \ldots, e_{\pi(k-n)}$ (each $e_i$ of which contains exactly one 1 at $i$-th position and the other elements are 0), $v_1, \ldots, v_n, e_{\pi(1)}, \ldots, e_{\pi(k-n)}$ are linearly independent. So one can take $V' = (0, \{v_1, \ldots, v_k\})$ and $v_1, \ldots, v_k$ are linearly independent. Then, it holds that: there exists a positive integer $\ell_V$, for $V'$ such that $\mathbb{N}^k = \{ u \mid \exists x \in \mathbb{N}, \exists y_1, \ldots, y_k \in \mathbb{Z} : 1 \leq x \leq \ell_V \text{ & } xu = \sum_{1 \leq i \leq k} y_i v_i \}$. First, we show that for each subset $I$ of $\{1, \ldots, k\}$, $S_I = \{ u \times (y_1, \ldots, y_k) \mid \exists x, y_1, \ldots, y_k \in \mathbb{N} : xu + \sum_{i \in I} y_i v_i = \sum_{j \not\in I} y_j v_j \}$ is effectively semi-linear. Note that $S_I \subseteq \mathbb{N}^{2k}$. Define the function $f_{I,x}$ of $\mathbb{N}^{2k}$ such that $f_{I,x}(p \times q) = (xp + \sum_{i \in I} q(i) v_i) \times \sum_{j \not\in I} q(j) v_j$, and define $g$ from $\mathbb{N}^k$ to $\mathbb{N}^{2k}$ such that $g(r) = r \times r$. Let $F = f_{I,x}((p_1 \times q_1) + (p_2 \times q_2))$, then

$$F = f_{I,x}((p_1 + p_2) \times (q_1 + q_2))$$

$$= ((xp_1 + \sum_{i \in I} q_1(i) v_i) + (xq_1 + \sum_{i \in I} q_2(i) v_i)) \times (\sum_{j \not\in I} q_1(i) v_i + \sum_{i \not\in I} q_2(i) v_i)$$

$$= ((xp_1 + \sum_{i \in I} q_1(i) v_i) \times \sum_{j \not\in I} q_1(i) v_i) + ((xq_1 + \sum_{i \in I} q_2(i) v_i) \times \sum_{i \not\in I} q_2(i) v_i)$$

$$= f_{I,x}(p_1 \times p_2) + f_{I}(q_1 \times q_2),$$

so $f_{I,x}$ is a linear function. Moreover, $g$ is a linear function, because $g(x + y) = (x + y) \times (x + y) = (x \times x) + (y \times y) = g(x) + g(y)$.

(Proof cont’d) 22
Appendix (B) : Complement of semi-linear sets (cont’d)

From (1), \( D = \{ g(p) \mid p \in \mathbb{N}^k \} \) is semi-linear. From (3), \( f_{I,x}^{-1}(D) \) is semi-linear. Observe that \( \bigcup_{1 \leq x \leq \ell_v} f_{I,x}^{-1}(D) = \bigcup_{1 \leq x \leq \ell_v} \{ p \in \mathbb{N}^{2k} \mid f_{I,x}(p) \in D \} = S_I. \)

Next, for each non-empty subset \( I \) of \( \{1, \ldots, k\} \), define \( c_I = (0, \ldots, 0, a_1, \ldots, a_k) \) where \( a_i = 1 \) if \( i \in I \); otherwise, \( a_i = 0 \). Let \( E_I = (c_I, \{ e_1, \ldots, e_{2k} \}) \) such that \( e_i \) (\( 1 \leq i \leq 2k \)) is the unit vector whose \( i \)-th element is 1, and let \( h(x \times y) = x \). For each \( I \subseteq \{1 \leq x \leq \ell_v\} \) with \( I \neq \emptyset \), define \( K_I = \bigcup_{1 \leq x \leq \ell_v} [[E_I]] \cap f_{I,x}^{-1}(D) \), then \( K_I \) is semi-linear, and thus, \( h(K_I) \) is semi-linear, because \( h \) is a linear function. We take \( T_I = h(K_I) \).

Since \( T_I = \{ u \in \mathbb{N}^k \mid \exists x, y_1, \ldots, y_k \in \mathbb{N} : xu = \sum_{i \in I} (-y_i)v_i + \sum_{j \notin I} y_jv_j \land y_i > 0 \ (i \in I) \}, \) \( T_I \cap [[V']] = \emptyset \) for all non-empty subset \( I \).

Next, define an NNVAS \( P_i \) (\( n < i \leq k \)) where \( P_i = (e_{n+i}, \{ e_1, \ldots, e_{2k} \}) \) and \( n \) is the number of periods of \( V \). Since \( [[P_i]] \) is semi-linear, \( [[P_i]] \cap K_I \) is semi-linear for each \( I \subseteq \{1, \ldots, k\} - \{i\} \). Let \( U_I = \bigcup_{n < i \leq k} h([[P_i]] \cap K_I). \) If \( I \subseteq \{1, \ldots, k\} - \{i\} \), then \( U_I = \{ u \in \mathbb{N}^k \mid \exists x, j \in \mathbb{N}, \exists y_1, \ldots, y_k \in \mathbb{Z} : xu = \sum_{i \in I} y_i v_i \land y_j > 0 \land j > n \}. \) Hence, \( U_I \cap [[V']] = \emptyset. \)

Next, for each \( x \ (1 \leq x \leq \ell_V) \) and \( j \ (1 \leq j \leq n) \), where \( n \) is the number of periods of \( V \), let
\[
Q_{x,j} = \{ u \times y \mid \exists y \in \mathbb{N}^n : xu = \sum_{1 \leq i \leq n} y(i)v_i \land y(j) \mod x \neq 0 \}
\]
\[
R_x = \{ u \times y \mid \exists y \in \mathbb{N}^n : xu = \sum_{1 \leq i \leq n} y(i)v_i \land y \neq 0 \}.
\]

We show that for every \( x \) and \( j \), \( Q_{x,j} \) is effectively semi-linear. The set \( \min \geq (R_x) \) of minimal solutions of \( R_x \) is finite and computable (Appendix (C)). (Proof cont’d)
Appendix (B) : Complement of semi-linear sets (cont’d)

Observe that $R_x = \llbracket (0, \min_{\geq}(R_x)) \rrbracket$. Moreover,

$$Q_{x,j} = R_x \cap \{ p \times y | \exists p \in \mathbb{N}^k, y \in \mathbb{N}^n : 1 \leq y(j) < x \}$$

Since $\{ p \times y | \exists p \in \mathbb{N}^k, y \in \mathbb{N}^n : 1 \leq y(j) < x \}$ is effectively semi-linear, $Q_{x,j}$ is so. Let $h'$ be the function from $\mathbb{N}^{k+n}$ to $\mathbb{N}^k$ such that $h'(p \times y) = p$, then $h'(Q_{x,j})$ is semi-linear, because $h'$ is a linear function.

Finally, we show that

$$([V'])^c = \bigcup_{\emptyset \neq I \subseteq \{1, \ldots, \ell_V\}} T_I \cup \bigcup_{1 \leq i \leq k, I \subseteq \{1, \ldots, k\} - \{i\}} U_I \cup \bigcup_{1 \leq x \leq \ell_V, 1 \leq j \leq n} h'(Q_{x,j}).$$

By construction, "$\supseteq$" is obvious: We already verified for the first two cases. For the last case, suppose $v \in Q_{x,j}$ for some $x (1 \leq x \leq \ell_V)$ and $j (1 \leq j \leq n)$, which means that $xv = \sum_{1 \leq i \leq n} y(i)v_i$ and $y(j)$ cannot be divided by $x$. This implies that if $x = 1$, $y(j)$ cannot be an integer. Hence, $v \notin [V']$.

For "$\subseteq$", suppose $v \in ([V'])^c$, then there exists $x \in \mathbb{N}$, $y_1, \ldots, y_k \in \mathbb{Z}$ such that $1 \leq x \leq \ell_V$ and $xv = \sum_{1 \leq i \leq k} y_i v_i$. If $y_i < 0$ ($1 \leq i \leq k$), then $v \in T_I$ for some non-empty subset $I$ of $\{1, \ldots, \ell_V\}$. If $y_i > 0$ ($n+1 \leq i \leq k$), then $v \in U_I$ for some $1 \leq j \leq k, I \subseteq \{1, \ldots, k\} - \{j\}$. So, assume $y_i \geq 0$ ($1 \leq i \leq n$). If for all $i$ ($1 \leq i \leq n$), $y_i$ is divided by $x$, $v \in [V']$. Thus, there exists $j$ such that $y_j$ is not divided by $x$. Hence, $v \in h'(Q_{x,j})$.

Alternative proof is obtained by bijective correspondence to Presburger arithmetic, where negation can be eliminated, and so negation-free NNVAS formula is obtained. 24
Appendix (C): Minimal solutions

Every semi-linear set \( \bigcup_{1 \leq i \leq n} \left\{ (c_i, \{v_{p_i(q_i)}\}) \right\} \) contains only finitely minimal elements \( c_i \) \((1 \leq i \leq n)\). This observation can be generalized as follows:

1. Every set of incomparable vectors in \( \mathbb{N}^k \) is finite.

**Proof**

Use the induction on \( k \). The base case is obvious, because \( k = 1 \). For induction step, define the projection \( f_k \) from \( \mathbb{N}^k \) to \( \mathbb{N}^{k-1} \) such that \( f_k(v) = (v(1), \ldots, v(k-1)) \). Suppose for leading to the contradiction that there exists an infinite subset \( S \) of \( \mathbb{N}^k \) whose elements are pairwise incomparable. For each \( u, v \in S \), one of the following holds: (a) \( f_k(u) \) and \( f_k(v) \) are incomparable, (b) \( f_k(u) > f_k(v) \) and \( u(k) < v(k) \), (c) \( f_k(u) < f_k(v) \) and \( u(k) > v(k) \). By induction hypothesis, (a) holds for only finitely many pairs. If (b) holds for infinitely many pairs, there exists an infinite sequence \( u_1, u_2, \ldots \) such that \( f_k(u_i) < f_k(u_{i+1}) \) and \( u_i(k) > u_{i+1}(k) \). However, it contradicts to the well-foundedness of \( > \) on \( \mathbb{N} \). For the same reason, (c) does not hold for infinitely many pairs, and hence, our assumption leads to the contradiction. \( \square \)

As a corollary of (1), it holds that: Every subset of \( \mathbb{N}^k \) contains only finitely many minimal elements.

In contrast, it holds that: If \( k \geq 2 \), for every subset of \( \mathbb{N}^k \) containing \( m \) minimal elements, there exists a subset of \( \mathbb{N}^k \) which contains more than \( m \) minimal elements (the number of incomparable minimal elements in \( \mathbb{N}^k \) \((k \geq 2) \) is unbounded).
(2) One can compute the set $S$ of minimal positive solutions of the equation:

$$w = \sum_{1 \leq i \leq m} x_i u_i - \sum_{1 \leq j \leq n} y_j v_j \quad (u_1, \ldots, u_m, v_1, \ldots, v_n \in \mathbb{N}^k, \ w \in \mathbb{Z}^k) \quad (*)$$

**Proof**

Let $V = \{u_i, v_j \mid 1 \leq i \leq m, \ 1 \leq j \leq n\}$. First, we show that the question if there exists a positive solution of $V$ is decidable. If $V \cup \{w\}$ is linear independent, there is no solution. If $V$ is linear independent and $V \cup \{w\}$ is linear dependent, then one can compute the a unique solution $p$ over $\mathbb{Q}^m$ and $q$ over $\mathbb{Q}^n$ such that $w = \sum_{1 \leq i \leq m} p(i) u_i - \sum_{1 \leq j \leq n} q(j) v_j$, which means that the equation has the positive solution over $\mathbb{N}^{m+n}$ if and only if $p \in \mathbb{N}^m$ and $q \in \mathbb{N}^n$. Suppose that $V$ is linear dependent. The following proof proceeds by induction on $m+n$. The base case (the case of $m+n = 1$) is obvious, because there is no such $V$ (the above equation forms $w = x_1 u_1$ or $w = -y_1 v_1$). For induction hypothesis, observe that one can compute subsets $I \subseteq \{1, \ldots, m\}$ and $J \subseteq \{1, \ldots, n\}$ and vectors $p \in \mathbb{N}^m$ and $q \in \mathbb{N}^n$ such that

$$\sum_{i \in I} p(i) u_i - \sum_{j \in J} q(j) v_j = \sum_{i \notin I} p(i) u_i - \sum_{j \notin J} q(j) v_j$$

with either $p(i) > 0$ for some $i \in I$ or $q(j) > 0$ for some $j \in J$. This implies that (*) has a positive solution $x_i, y_j$ ($1 \leq i \leq m, \ 1 \leq j \leq n$) if and only if it satisfies either $x_i \leq p(i)$ for some $i \in I$ or $y_j \leq q(j)$ for some $j \in J$. (Proof cont’d)
This is because if \((x_1, \ldots, x_m, y_1, \ldots, y_n) > (p \times q)\), then
\[
w = \sum_{i \in I} (x_i - p(i))u_i + \sum_{i \notin I} (x_i + p(i))u_i - \sum_{j \in J} (y_j - q(j))v_j - \sum_{j \notin J} (y_j + q(j))v_j
\]
such that \(x_i - p(i) < x_i\) for some \(i \in I\) or \(y_j - q(j) < y_j\) for some \(j \in J\). By repeating the above computation, we obtain a positive solution of \((*1)\) such that \(x_i \leq p(i)\) for some \(i \in I\) or \(y_j \leq q(j)\) for some \(j \in J\). Let \(X = \{(k, a) \mid a \in I, k \leq p(i)\}\) and \(Y = \{(\ell, b) \mid b \in J, \ell \leq q(j)\}\). Then, the equation \((*1)\) has a positive solution if and only if there exists \((k, a) \in X\) such that \(w - ku_a = \sum_{i \in I \setminus \{a\}} p(i)u_i - \sum_{j \in J} q(j)v_j\) \((*2)\) has a solution or \((\ell, b) \in Y\) such that \(w + \ell v_b = \sum_{i \in I} p(i)u_i - \sum_{j \in J \setminus \{b\}} q(j)v_j\) \((*3)\) has a solution. Here "\((*2)\) has a solution" means the equation \((*2)\) has a positive solution or the solution is 0 (where \(k > 0\) and the other \(p(i)\)'s are 0). Similar to \((*3)\). By induction hypothesis, the question if \((*2)\) or \((*3)\) has a solution is decidable. Hence, since \(X, Y\) are finite, the question if \((*1)\) has a positive solution is decidable.

Next, we show our statement. According to the above observation, one can determine if there is a positive solution of \((*1)\). If there is no solution, the empty set is the answer. Otherwise, one can find a positive solution of \((*1)\). Since the number of vectors smaller than the solution is finite, one can find a minimal positive solution of \((*1)\), say \(s\). If there is another minimal positive solution of \((*1)\), say \(t\), then for some \(c, d \in \{1, \ldots, m+n\}\), \(t(c) < s(c)\) and \(t(d) > s(d)\). So, if \(1 \leq c \leq m\), consider the equation \((*2)\) where \(k = c\) and \(u_a = t(c)\). (Proof cont'd)
Similarly, if $m < c \leq n$, consider the equation (*3) where $\ell = c$ and $v_b = t(c)$. Since the number of the candidates for the pairs of such $(c, t(c))$ is finite, by induction hypothesis, one can compute the set $S_c$ of minimal positive solutions of (*2) and the set $T_c$ of minimal positive solutions of (*3). Let $f_{x,c}$ be the function from $\mathbb{N}^{m+n-1}$ to $\mathbb{N}^{m+n}$ such that $f_{x,c}(w) = (w(1), \ldots, w(c-1), x, w(c), \ldots, w(m+n-1))$. Then, $S = \min_{x}(\bigcup_{1 \leq c \leq m+n} \bigcup_{0 \leq x \leq s(c)} \{f_{x,c}(w) \mid w \in S_c \cup T_c\})$. Hence, we can compute the set of minimal positive solutions of (*1).