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# Time-scaling Control of an Underactuated Manipulator

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## Abstract

*Position control of an underactuated manipulator that has one passive joint is investigated. The dynamic constraint caused by the passive joint is second-order nonholonomic. Time-scaling of the active joint trajectory and bi-directional motion planning from the initial and the desired configurations provide an exact solution of the positioning trajectory. The active and passive joints can be positioned to the desired angles simultaneously by swinging the active joints only twice. Feedback control constrains the manipulator along the planned path in the configuration space. Simulation and experimental results show the validity of the proposed methods.*

**Key Words:** *Underactuated Manipulator, Passive Joint, Nonholonomic Constraint, Time-scaling*

## 1 Introduction

Given a class of robotic manipulation tasks, it might be possible to use a simpler robot mechanism (e.g. fewer joints, actuators or sensors) than ordinary ones to perform the task by considering and utilizing the task dynamics. Control of underactuated mechanisms, which have fewer actuators than the number of the generalized coordinates associated with the task, has received increasing attention from the viewpoint of nonholonomic systems in recent years. For example, if we can dexterously control an underactuated manipulator that has passive joints equipped with no actuators, the weight, cost and energy consumption can be reduced, and failure recovery of even fully-actuated manipulators can be facilitated.

An underactuated manipulator is under a dynamic constraint caused by the zero torque at the passive joints. The constraint is generally a nonholonomic constraint as nonintegrable differential equations, unless the joints are placed in some special way. Exploiting this constraint, the positioning of the manipulator to the desired configuration can be achieved by the motions of the active joints, even if the passive joints are completely free joints with no brakes, etc.

The nonholonomic constraint of the underactuated manipulator has different characteristics from those of wheeled vehicles and space robots, which have been treated as typical nonholonomic systems. The constraints of these examples are caused by the rolling contact or the conservation of angular momentum, and are represented as a first-order nonintegrable differential equation, $\boldsymbol{H}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0}$, where $\boldsymbol{q}$ is the generalized coordinate and $\dot{\boldsymbol{q}}$ is the generalized velocity. The state equation is written as a drift-free symmetrical affine system, $\dot{\boldsymbol{q}} = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{u}$, with the velocity input $\boldsymbol{u}$. On the other hand, the dynamic constraint of the underactuated manipulator is described as a second-order nonintegrable differential equation, $\boldsymbol{M}_p(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{b}_p(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{0}$, including the generalized acceleration $\ddot{\boldsymbol{q}}$, and is called a second-order nonholonomic constraint. The state equation, $\frac{d}{dt}[\boldsymbol{q}^T, \dot{\boldsymbol{q}}^T]^T = \boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{G}(\boldsymbol{q})\boldsymbol{u}$, with the acceleration or torque input $\boldsymbol{u}$, has a drift term $\boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}})$.

There have been many studies on the conversion of symmetrical affine systems to standard forms such as the chained system or Caplygin system, and also on motion planning and feedback control based upon those forms. However, those methods cannot be directly applied to the affine system with a drift term. Thus there are no unified control methods for underactuated manipulators yet, and most of the methods so far proposed rely on the specific dynamics of the individual mechanisms.

Here, we review the previous studies which treat the underactuated manipulators as nonholonomic systems. Oriolo and Nakamura [1] derived the condition when the constraint by the passive joints is non-integrable and hence nonholonomic. They also proved that an underactuated manipulator cannot be stabilized to an equilibrium point by using any smooth state feedback when no potential force such as gravity is applied to the passive joints. The first linear approximation of such a system is not controllable. In contrast, the linear approximation of an underactuated mechanism in the gravity field, e.g. an inverted pendulum, is controllable and can be stabilized to the equilibrium point with smooth state feedback, although such a mechanism is also second-order nonholonomic.

A two-axis horizontal manipulator in which the first joint is active and the second joint is passive is the simplest articulated form of an underactuated manipulator. Ref. [1] showed that its constraint is second-order nonholonomic unless the center of mass of the free link coincides with the passive joint. De Luca et al. [4] proved this manipulator is locally accessible but does not satisfy the sufficient condition for small-time local controllability in Ref. [14]. The global controllability has not been proved analytically.

Suzuki et al. [2] stabilized the passive joint angle of this manipulator to an arbitrary angle by a periodical motion of the active joint. They described the behavior of the passive joint as a Poincaré map and modified the amplitude of the active joint motion according to the angle and angular velocity of the passive joint. Suzuki et al. [3] used an averaging method to represent the system in a simpler form and designed the feedback control using a Lyapunov function. De Luca et al. [4] proposed the position control by repeating the open-loop control, by which the system approaches the desired state, based on the nilpotent approximation.

There have also been studies on the control of a horizontal three-axis underactuated manipulator in which the third joint is passive. Ref. [5, 6] showed the small-time local controllability of the equivalent mechanism at the equilibrium point. We proved the global controllability of this manipulator by considering the behavior of the center of percussion of the free link, and constructing the trajectory between two arbitrary states [7]. We also proposed a feedback control to stabilize the translational and rotational trajectory segments in the positioning trajectory [8]. Imura et al. [9] derived the second-order chained form from the link dynamics by transformation of the coordinates and inputs. They also proposed a time-varying and non-continuous feedback control which asymptotically stabilizes the system from an arbitrary initial state to the desired configuration.

Rathinam and Murray [10] showed the condition for configuration flatness (the system is differentially flat and the flat output depends only on the generalized coordinates) of a mechanical system with one passive joint. They also presented the method to calculate the flat output when it exists. If the system is configuration flat (e.g. the three-axis underactuated manipulator in [7, 8, 9]), the possible trajectory between two states can be systematically constructed using the flat output.

Here, we propose a motion planning and feedback control method for an underactuated manipulator which has one passive joint with no gravity applied. Ref. [2, 3, 4, 7, 8, 9] already proposed control methods for manipulators belonging to this class. However, the methods in [2, 3, 4] resort to repetitive motion of the active joint and usually take a long time for the positioning. The methods in [7, 8, 9] can be applied only when the passive joint is the final axis and can move freely in the horizontal plane. Though Ref. [10] probides a general and systematic approach, it cannot be used unless the system has the flat output. For example, the two-axis manipulator in [2, 3, 4] is not differentially flat. In this paper, the trajectory for the positioning is planned by time-scaling of the active joint trajectory and bi-directional planning from the initial and final configurations. This method can provide an exact solution (not an approximated one) for the position control. The manipulator can reach the desired configuration by swinging the active joints only twice.

The rest of this paper is organized as follows. In Section 2, the manipulator is modeled to show the dynamic constraint caused by the passive joint. In Section 3, the behavior of the manipulator is considered when the time-axis of the active joint trajectory stretches or shrinks. Then the trajectory for the positioning is designed by the bi-directional planning from the initial and the desired configurations. The feedback control for tracking of the desired path is proposed in Section 4. The simulations and experiments in Section 5 demonstrate that the manipulator can be positioned to the desired configuration by the proposed method.

## 2 Model of Manipulator

We consider an $n$-axis serial underactuated manipulator which has $n$-1 active joints and one passive joint. The passive joint is a revolute joint without angle limit, and neither gravity nor friction torque acts on it. A horizontal underactuated manipulator with a passive revolute joint around a vertical axis is a typical example. We define $\boldsymbol{\theta}_a = [\theta_1, ..., \theta_{n-1}]^T \in \Re^{n-1}$ as the active joint angle and $\theta_p = \theta_n \in \Re$ as the passive joint angle. The equation of motion of the manipulator is:

$$\begin{aligned} \boldsymbol{M}_{aa}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}}_a + \boldsymbol{m}_{pa}^T(\boldsymbol{\theta})\ddot{\theta}_p + \boldsymbol{b}_a(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) &= \boldsymbol{\tau}_a \quad (1) \\ \boldsymbol{m}_{pa}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}}_a + m_{pp}(\boldsymbol{\theta})\ddot{\theta}_p + b_p(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) &= 0. \quad (2) \end{aligned}$$

The passive joint is not necessarily at the $n$-th axis. The elements of the vectors and matrices can be rearranged to get the above representation. Eq.(2) means the dynamic constraint caused by the zero torque at the passive joint.

$$\boldsymbol{M}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{M}_{aa}(\boldsymbol{\theta}) & \boldsymbol{m}_{pa}^T(\boldsymbol{\theta}) \\ \boldsymbol{m}_{pa}(\boldsymbol{\theta}) & m_{pp}(\boldsymbol{\theta}) \end{bmatrix}$$

$$(\boldsymbol{M}_{aa} \in \Re^{(n-1)\times(n-1)}, \boldsymbol{m}_{pa} \in \Re^{n-1}, m_{pp} \in \Re)$$

denotes the inertia matrix.

$$\boldsymbol{b}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \begin{bmatrix} \boldsymbol{b}_a(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \\ b_p(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \end{bmatrix} \quad (\boldsymbol{b}_a \in \Re^{n-1}, b_p \in \Re)$$

is a Coriolis and centrifugal term, and generally has a form of

$$\begin{aligned} \boldsymbol{b}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) &= \dot{\boldsymbol{M}}\dot{\boldsymbol{\theta}} - [\frac{1}{2}\dot{\boldsymbol{\theta}}^T \frac{\partial \boldsymbol{M}}{\partial \theta_1}\dot{\boldsymbol{\theta}}, ..., \frac{1}{2}\dot{\boldsymbol{\theta}}^T \frac{\partial \boldsymbol{M}}{\partial \theta_n}\dot{\boldsymbol{\theta}}]^T \\ &= \sum_{j=1}^{n}\sum_{k=1}^{j} \boldsymbol{c}_{jk}(\boldsymbol{\theta})\dot{\theta}_j\dot{\theta}_k. \quad (3) \end{aligned}$$

This manipulator is assumed not to satisfy the condition for integrability in Ref. [1]. That is, the inertia matrix $\boldsymbol{M}(\boldsymbol{\theta})$ explicitly includes the passive joint angle $\theta_p$. Then the constraint (2) does not have the first integral, which is a function of the joint angle $\boldsymbol{\theta}$ and the angular velocity $\dot{\boldsymbol{\theta}}$, and is a second-order nonholonomic constraint that includes the angular acceleration $\ddot{\boldsymbol{\theta}}$.

Since $m_{pp}(\boldsymbol{\theta}) \neq 0$ from the property of the inertia matrix, the angular acceleration $\ddot{\theta}_p$ of the passive joint is,

$$\ddot{\theta}_p = -m_{pp}(\boldsymbol{\theta})^{-1}\{\boldsymbol{m}_{pa}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}}_a + b_p(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\} \quad (4)$$

from Eq.(2). The angular acceleration of the active joints is treated as the control input $\boldsymbol{u} = \ddot{\boldsymbol{\theta}}_a$. The state equation can be represented as an affine system,

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{\theta}_a \\ \theta_p \\ \dot{\boldsymbol{\theta}}_a \\ \dot{\theta}_p \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{\theta}}_a \\ \dot{\theta}_p \\ \boldsymbol{0} \\ -m_{pp}^{-1}b_p \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{I} \\ -m_{pp}^{-1}\boldsymbol{m}_{pa} \end{bmatrix}\boldsymbol{u}. \quad (5)$$

The first term in the right side is the drift term.

# 3 Motion Planning

We develop a motion planning method for the position control in this section. Namely, we find the trajectory and input to transfer the manipulator from the initial configuration to the desired configuration. The manipulator should have zero velocity at the initial and final state. Though it is easy to stop the active joints, the passive joint usually continues moving due to the drift. Therefore, we must choose the trajectory along which the active and passive joints simultaneously stop at the desired angles.

First, we discuss the behavior of the manipulator when the trajectory profile of the active joints is scaled along the time-axis. Next, we consider the trajectory that can be realized from the initial state, and the trajectory that can reach the desired state. Those two trajectories are connected by the free rotation of the passive joint. The idea of bi-directional trajectory planning from both the initial and desired states was suggested by Ref. [11], which proposed a bi-directional approach to the motion planning of a space robot. Our method gives an exact solution, not an approximated one, for the positioning. In the planned trajectory, all the joints can be positioned simultaneously to the desired angles by swinging the active joints only twice.

## 3.1 Time-scaling of Trajectory

Here, we consider the motion of the passive joint when the time-axis of the active joint trajectory stretches or shrinks uniformly. It is shown that the angle of the passive joint at the final point of the trajectory does not vary however the time-axis is scaled, and that the angular velocities of the passive joint at both ends of the trajectory are proportional to the scaling factor.

The active joints are given the following trajectory,

$$\boldsymbol{\theta}_a = \boldsymbol{f}(\kappa t) = [f_1(\kappa t), ..., f_{n-1}(\kappa t)]^T, \quad (6)$$

which is a function of time $t$ $(0 \leq t \leq 1/\kappa)$. $f_1(\cdot), ..., f_{n-1}(\cdot)$ are twice differentiable scalar functions. The constant $\kappa > 0$ is the time-scaling factor. Suppose $s = \kappa t$ is a new time-variable, then

$$\dot{\theta}_i = \kappa\frac{d\theta_i}{ds}, \quad \ddot{\theta}_i = \kappa^2\frac{d^2\theta_i}{ds^2} \quad (i = 1, ..., n). \quad (7)$$

Substituting Eq.(7) to Eq.(4), and considering Eq.(3), we obtain

$$\frac{d^2\theta_p}{ds^2} = -m_{pp}(\boldsymbol{\theta})^{-1}\{\boldsymbol{m}_{pa}(\boldsymbol{\theta})\frac{d^2\boldsymbol{\theta}_a}{ds^2} + b_p(\boldsymbol{\theta}, \frac{d\boldsymbol{\theta}}{ds})\}. \quad (8)$$

Eq.(8) is a differential equation with $s$ being the independent variable and includes neither $t$ nor $\kappa$ explicitly. The input, $d^2\boldsymbol{\theta}_a/ds^2$, is the second-order derivative of $\boldsymbol{f}(s)$ with regard to $s$, and depends on $s$ only. Hence the solution of this differential equation can be represented as a function of $s$, $\theta_p(s)$. In other words, the manipulator moves along the same path [1] irrespective of $\kappa$. When the passive joint starts from the same initial state $\theta_p|_{s=0}, d\theta_p/ds|_{s=0}$ and the active joints are given the same trajectory $\boldsymbol{\theta}_a = \boldsymbol{f}(s)$ $(0 \le s \le 1)$, the passive joint reaches the same final state $\theta_p|_{s=1}, d\theta_p/ds|_{s=1}$. Thus $\theta_p$ and $\dot{\theta}_p/\kappa$ at $t = 1/\kappa$ are independent of the scaling factor $\kappa$ and have constant values.

Furthermore, we add the following boundary condition,

$$\boldsymbol{f}(0) = \boldsymbol{\theta}_{a0}, \boldsymbol{f}(1) = \boldsymbol{\theta}_{a1}, \frac{d\boldsymbol{f}}{ds}(0) = \frac{d\boldsymbol{f}}{ds}(1) = \boldsymbol{0}$$

on the active joint trajectory $\boldsymbol{f}(s)$. The active joints start from $\boldsymbol{\theta}_{a0}$ with zero velocity, and stop again at $\boldsymbol{\theta}_{a1}$ for any time-scaling factor $\kappa$. We call such a motion as a "swing" of the active joints. Suppose the passive joint takes the initial state $(s = 0)$ as zero velocity at the angle $\theta_{p0}$. The active joints stop at $s = 1$, but the passive joint does not stop due to the drift term in Eq.(5). It usually has non-zero velocity $\dot{\theta}_{p1}$ at the angle $\theta_{p1}$. Those values can be calculated by giving the active joint acceleration $\ddot{\boldsymbol{\theta}}_a$ to Eq.(4) and integrating the equation numerically from the initial state. The passive joint has the same angle $\theta_{p1}$ at the end of the trajectory for arbitrary scaling factor $\kappa$, while the angular velocity $\dot{\theta}_{p1}$ is proportional to $\kappa$.

Conversely, we can consider such angle $\theta_{p0}$ and angular velocity $\dot{\theta}_{p0}$ at $s = 0$ that the passive joint has zero velocity at the end of the swing $(s = 1)$. $\theta_{p0}$ and $\dot{\theta}_{p0}$ can be calculated backwards by integrating Eq.(4) in the reverse direction from $t = 1/\kappa$ to $t = 0$. Such a trajectory can exist unless $d^2\theta_p/ds^2 \equiv 0$ during the swing. The passive joint angle $\theta_{p0}$ at the initial state does not change regardless of the scaling factor $\kappa$, and the angular velocity $\dot{\theta}_{p0}$ is proportional to $\kappa$.

### 3.2 Bi-directional Motion Planning

The trajectory for the positioning between two configurations is planned utilizing the property of the time-scaling described above. It is assumed that the active joints stop at a certain intermediate position besides the initial and the desired positions. The positioning trajectory is composed of three trajectory primitives. The first one is the trajectory from the initial state which can be realized by the swing of the active joints from the initial to the intermediate position. The next one is the free rotation of the passive joint while the active joints stay at the intermediate position. The last one is the trajectory that can reach the desired state by the swing from the intermediate to the desired position. Those trajectories are parameterized by the time-scaling factor for each swing and the intermediate position of the active joints. The parameters of the swings are chosen so that the trajectory from the initial state and the trajectory to the desired state can be smoothly connected by the trajectory of free rotation.

The initial and the desired configurations of the manipulator are denoted by $[\boldsymbol{\theta}_{a0}^T, \theta_{p0}]^T$ and $[\boldsymbol{\theta}_{ad}^T, \theta_{pd}]^T$, respectively. The intermediate position of the active joints is represented as $\boldsymbol{\theta}_{am}$. First, the active joints move from the initial position $\boldsymbol{\theta}_{a0}$ to the intermediate position $\boldsymbol{\theta}_{am}$ according to the trajectory

$$\boldsymbol{\theta}_a = \boldsymbol{f}_A(\kappa_A t) \quad (0 \le t \le 1/\kappa_A)$$

which satisfies the following boundary conditions,

$$\boldsymbol{f}_A(0) = \boldsymbol{\theta}_{a0}, \boldsymbol{f}_A(1) = \boldsymbol{\theta}_{am}, \frac{d\boldsymbol{f}_A}{ds}(0) = \frac{d\boldsymbol{f}_A}{ds}(1) = \boldsymbol{0}$$

(Swing A). Then the active joints stay there for a while. Finally, the active joints move to the desired position $\boldsymbol{\theta}_{ad}$ according to the trajectory

$$\boldsymbol{\theta}_a = \boldsymbol{f}_B(\kappa_B t) \quad (0 \le t \le 1/\kappa_B)$$

satisfying the boundary conditions,

$$\boldsymbol{f}_B(0) = \boldsymbol{\theta}_{am}, \boldsymbol{f}_B(1) = \boldsymbol{\theta}_{ad}, \frac{d\boldsymbol{f}_B}{ds}(0) = \frac{d\boldsymbol{f}_B}{ds}(1) = \boldsymbol{0}$$

(Swing B). **Figure 1** and **Figure 2** show the trajectories of the active and passive joints in the state space for a two-axis manipulator.

When the time-scaling factor $\kappa_A$ for Swing A equals 1, the angle and angular velocity of the passive joint at the instant the active joints reach $\boldsymbol{\theta}_{am}$ are denoted by $\theta_{pA}^{\kappa_A=1}$ and $\dot{\theta}_{pA}^{\kappa_A=1}$, respectively. These values can be easily obtained by numerically integrating Eq.(8) over $0 \le s \le 1$. In the same way, the time-scaling factor $\kappa_B$ for Swing B is assumed to be 1. Then the initial passive joint angle $\theta_{pB}^{\kappa_B=1}$ and the angular velocity $\dot{\theta}_{pB}^{\kappa_B=1}$ at which the manipulator stops at the desired configuration $\boldsymbol{\theta}_{ad}$, $\theta_{pd}$ after Swing B can be calculated

[1] A *path* means a geometrical curve in the configuration space that is not associated with time. We make a distinction from a *trajectory*, which represents the motion with the progress of time.

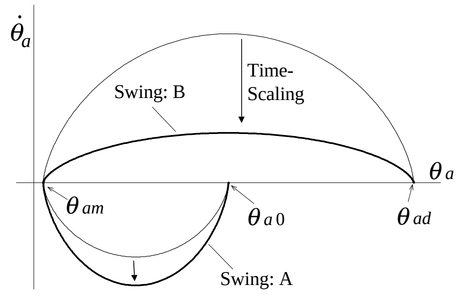


Fig. 1: Trajectory of active joint

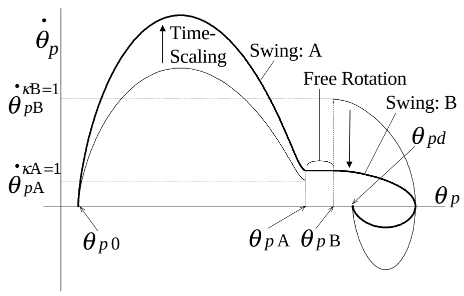


Fig. 2: Trajectory of passive joint

back by the integration of Eq.(8), where the time-axis is reversed in this case. See the thin lines of **Figure 1** and **Figure 2**.

When the time-scaling factor $\kappa_A$ for Swing A has an arbitrary positive value, the passive joint angle $\theta_{pA}$ at the end of the swing does not depend on $\kappa_A$ and is constant. The angular velocity $\dot{\theta}_{pA}$ is proportional to $\kappa_A$.

$$\theta_{pA} = \theta_{pA}^{\kappa_A=1}, \quad \dot{\theta}_{pA} = \kappa_A \dot{\theta}_{pA}^{\kappa_A=1} \tag{9}$$

The passive joint angle $\theta_{pB}$ at the start of Swing B does not change with $\kappa_B$, and the angular velocity $\dot{\theta}_{pB}$ is proportional to $\kappa_B$, too.

$$\theta_{pB} = \theta_{pB}^{\kappa_B=1}, \quad \dot{\theta}_{pB} = \kappa_B \dot{\theta}_{pB}^{\kappa_B=1} \tag{10}$$

While the active joints stand still at the intermediate position, the manipulator is equivalent to a rigid link pivoted around the passive joint at a fixed location in the inertial coordinate frame. Therefore the passive joint rotates with constant angular velocity. The passive joint passes through $\theta_{pB}$ if $\dot{\theta}_{pA} \neq 0$, since the passive joint is a limitless revolute axis. If $\dot{\theta}_{pA} = \dot{\theta}_{pB}$ and Swing B starts at the moment the passive joint reaches $\theta_{pB}$, Swing A and B can be smoothly connected. The angular velocity of the passive joint at the end of Swing A and the start of Swing B can be equalized by the time-scaling, if we can find such intermediate position of the active joints, $\boldsymbol{\theta}_{am}$, that $\dot{\theta}_{pA}^{\kappa_A=1}$ and $\dot{\theta}_{pB}^{\kappa_B=1}$ have the same turning directions. From Eq.(9) and (10), the time-scaling factors should satisfy,

$$\kappa_A / \kappa_B = \dot{\theta}_{pB}^{\kappa_B=1} / \dot{\theta}_{pA}^{\kappa_A=1}. \tag{11}$$

The time-axis of the active joint trajectory in each swing is scaled using these $\kappa_A$ and $\kappa_B$, like the thick line in **Figure 1**. Then the state of the passive joint traces the thick line in **Figure 2** and stops at the desired angle. The pause at the intermediate position $\boldsymbol{\theta}_{am}$ is the smallest positive value of $(\theta_{pB} - \theta_{pA} + 2n\pi)/\dot{\theta}_{pA}$, where $n$ is an integer.

The necessary condition to plan the desired trajectory with the above method is that there exists such intermediate position, $\boldsymbol{\theta}_{am}$, that $\dot{\theta}_{pA}^{\kappa_A=1}$ and $\dot{\theta}_{pB}^{\kappa_B=1}$ have the same signs. It is neither obvious nor easy to show analytically for what combination of the initial and the desired configurations there exists such $\boldsymbol{\theta}_{am}$. Instead, we numerically show later that such intermediate positions exist in a wide area for a two-axis planar manipulator.

Provided the above condition is satisfied, the intermediate position $\boldsymbol{\theta}_{am}$, the angular velocity of the passive joint during the free rotation, and the trajectory profile of the active joints $\boldsymbol{f}_A(\cdot)$, $\boldsymbol{f}_B(\cdot)$ remain unconstrained as the freedoms in the trajectory design, and can be exploited for optimizing the trajectory (e.g. minimum-time).

Most of the computational effort to plan the motion lies in the numerical integration of Eq.(8) for Swing A and B. One set of the integrations for one intermediate position $\boldsymbol{\theta}_{am}$ provides information on the possibility of positioning (whether $\dot{\theta}_{pA}^{\kappa_A=1}$ and $\dot{\theta}_{pB}^{\kappa_B=1}$ have the same sign) and the ratio of the time-scaling factors, $\kappa_A/\kappa_B$. The computational load depends on the search method for $\boldsymbol{\theta}_{am}$ and is generally small. Once $\boldsymbol{\theta}_{am}$ that enables the positioning is found, the exact solution trajectory is obtained immediately. Note that the solution is not an approximated trajectory by iterative calculations.

## 4 Feedback Control

Since no potential force such as gravity acts on the passive joint, this manipulator cannot be asymptotically stabilized to an equilibrium point with any smooth state feedback [1]. However, the open-loop trajectory for the positioning from the initial configuration to the desired configuration is rigorously calculated by the motion planning in the previous section.

The manipulator is expected to reach the neighborhood of the desired configuration if some feedback control can be assembled in accordance with this nominal trajectory.

Actually, the passive joint angle $\theta_{pA}$ and the angular velocity $\dot{\theta}_{pA}$ just after Swing A do not have to completely coincide with the planned values, as far as $\dot{\theta}_{pA}$ has the same sign as the angular velocity $\dot{\theta}_{pB}$ at the start of Swing B. The time-scaling factor $\kappa_B$ for Swing B is modified as $\kappa_B = \dot{\theta}_{pB}/\dot{\theta}_{pB}^{\kappa_B=1}$, where $\dot{\theta}_{pB}$ is the measured angular velocity when the passive joint reaches $\theta_{pB}$. The trajectory of Swing B that stops at the desired configuration can be renewed using this $\kappa_B$. Consequently, feedback control is essential in Swing B.

The manipulator moves along the geometrically identical path in the configuration space when the time-axis of the active joint trajectory is scaled uniformly. If the posture of the manipulator is constrained along the desired path, the time-axis might be automatically scaled and the velocity profile along the path could be what the desired trajectory is scaled to be. Since the desired trajectory is planned so that the velocity of each joint becomes zero at both ends of the trajectory, the manipulator is expected to stop at the desired configuration however the time-axis is scaled.

In this section, we first show that the swing of $n-1$ active joints can be represented by one coordinate. Next, the feedback control is developed to make the manipulator follow the desired path using the acceleration of this coordinate as the input. We proposed the path tracking control of an underactuated manipulator in Ref.[12]. However, in that method the path coordinate frame had to be defined based on the desired path. The feedback control law in this paper does not use the path coordinate frame and requires the planned trajectory data only.

### 4.1 Single Coordinate Representation of Active Joints

The motions of the $n-1$ active joints are constrained by each other according to the function $\boldsymbol{f}(\cdot)$ as in Eq.(6). These motions can be combined to one degree-of-freedom motion. Suppose the coordinate which represents this motion is $x_a \in \Re$, then the active joint position $\boldsymbol{\theta}_a$ is represented as,

$$\boldsymbol{\theta}_a = \boldsymbol{g}(x_a) = [g_1(x_a), ..., g_{n-1}(x_a)]^T. \qquad (12)$$

$x_a$ can be considered as a distance along the path of the active joints. The function $\boldsymbol{g}(\cdot)$ means the coordinate transformation from $x_a$ to $\boldsymbol{\theta}_a$. The active joint velocity $\dot{\boldsymbol{\theta}}_a$, and acceleration $\ddot{\boldsymbol{\theta}}_a$ are

$$\dot{\boldsymbol{\theta}}_a = \frac{d\boldsymbol{g}}{dx_a}\dot{x}_a, \quad \ddot{\boldsymbol{\theta}}_a = \frac{d^2\boldsymbol{g}}{dx_a^2}\dot{x}_a^2 + \frac{d\boldsymbol{g}}{dx_a}\ddot{x}_a. \qquad (13)$$

Substituting them to the constraint (2) results in

$$\boldsymbol{m}_{pa}\frac{d\boldsymbol{g}}{dx_a}\ddot{x}_a + m_{pp}\ddot{\theta}_p + b_p + \boldsymbol{m}_{pa}\frac{d^2\boldsymbol{g}}{dx_a^2}\dot{x}_a^2 = 0.$$

This equation can be rewritten as

$$m_a(x_a, \theta_p)\ddot{x}_a + m_p(x_a, \theta_p)\ddot{\theta}_p + b(x_a, \theta_p, \dot{x}_a, \dot{\theta}_p) = 0 \qquad (14)$$

where,

$$m_a = \boldsymbol{m}_{pa}\frac{d\boldsymbol{g}}{dx_a}, \quad m_p = m_{pp}, \quad b = b_p + \boldsymbol{m}_{pa}\frac{d^2\boldsymbol{g}}{dx_a^2}\dot{x}_a^2. \qquad (15)$$

Eq.(14) represents the dynamic constraint between the coordinate $x_a$ describing the active joint motion, and the passive joint angle $\theta_p$. It has the same form as the constraint of a manipulator with one active and one passive joint.

### 4.2 Feedback for Path Tracking

We assume the desired geometrical path in the configuration space is represented as

$$\theta_p = \theta_{pd}(x_a). \qquad (16)$$

In other words, the passive joint angle $\theta_p$ is uniquely determined for the position of the active joints, $x_a$. Eq.(16) defines the geometrical relationship between $\theta_p$ and $x_a$, and does not depend on time. This relationship can be obtained by eliminating $s$ from the desired trajectory

$$\theta_p = \theta_{pd}(s), \quad x_a = x_{ad}(s)$$

planned for each swing. $x_a$ should be monotonously increasing or decreasing, and $x_a$ and $s$ should have a one-to-one correspondence, in order to represent the path as Eq.(16). $x_a$ is not allowed to stop or go back along the trajectory. Such swings of the active joints must be given in the motion planning. The angle error $e_p$ of the passive joint from the desired path is defined as

$$e_p = \theta_p - \theta_{pd}(x_a). \qquad (17)$$

Differentiating the above equation with respect to time,

$$\dot{e}_p = \dot{\theta}_p - \frac{d\theta_{pd}}{dx_a}\dot{x}_a \qquad (18)$$

$$\ddot{e}_p = \ddot{\theta}_p - \frac{d\theta_{pd}}{dx_a}\ddot{x}_a - \frac{d^2\theta_{pd}}{dx_a^2}\dot{x}_a^2 \quad (19)$$

The angular acceleration of the passive joint, $\ddot{\theta}_p$, is,

$$\ddot{\theta}_p = -m_p^{-1}(m_a\ddot{x}_a + b)$$

from the dynamic constraint (14). Substituting it to Eq.(19),

$$\ddot{e}_p = -(m_p^{-1}m_a + \frac{d\theta_{pd}}{dx_a})\ddot{x}_a - m_p^{-1}b - \frac{d^2\theta_{pd}}{dx_a^2}\dot{x}_a^2 \quad (20)$$

When the acceleration of the error is $\ddot{e}_p$, the acceleration of the active joints is,

$$\ddot{x}_a = -\frac{\ddot{e}_p}{m_p^{-1}m_a + \frac{d\theta_{pd}}{dx_a}} - \frac{m_p^{-1}b + \frac{d^2\theta_{pd}}{dx_a^2}\dot{x}_a^2}{m_p^{-1}m_a + \frac{d\theta_{pd}}{dx_a}} \quad (21)$$

if $m_p^{-1}m_a + \frac{d\theta_{pd}}{dx_a} \neq 0$. $\ddot{e}_p$ is determined by the following PD feedback,

$$\ddot{e}_p = -k_V\dot{e}_p - k_P e_p \quad (22)$$

where $k_P$ and $k_V$ are position and velocity feedback gains, respectively. The acceleration $\ddot{x}_a$ of the active joints is calculated by substituting Eq.(22) to Eq.(21). Then the error converges to zero as $\ddot{e}_p + k_V\dot{e}_p + k_P e_p = 0$. The manipulator is thus constrained to the desired path.

The second term of the right side in Eq.(21) is the feedforward term in the sense that it gives the motion along the desired path if $e_p = 0$. However, it is actually a state feedback which does not depend on time, since it consists of the functions of $x_a$, $\theta_p$, $\dot{x}_a$ and $\dot{\theta}_p$ only. The acceleration of the active joints is not given as a function of time.

The calculations of Eq.(17), (18), (21) and (22) require $\theta_{pd}(x_a)$, $\frac{d\theta_{pd}}{dx_a}(x_a)$ and $\frac{d^2\theta_{pd}}{dx_a^2}(x_a)$ as functions of $x_a$ in real-time. $\theta_{pd}(s)$, $\frac{d\theta_{pd}}{ds}(s)$ and $\frac{d^2\theta_{pd}}{ds^2}(s)$ can be obtained in advance as functions of $s$ from Eq.(8) and its integration, while the desired trajectory is planned. $x_a(s)$, $\frac{dx_a}{ds}(s)$ and $\frac{d^2x_a}{ds^2}(s)$ are calculated as the active joint motion. If these values are stored in memory arrays, $\frac{d\theta_{pd}}{dx_a}(s)$ and $\frac{d^2\theta_{pd}}{dx_a^2}(s)$ can be calculated as follows:

$$\frac{d\theta_{pd}}{dx_a} = \frac{\frac{d\theta_{pd}}{ds}}{\frac{dx_a}{ds}}, \quad \frac{d^2\theta_{pd}}{dx_a^2} = \frac{\frac{d^2\theta_{pd}}{ds^2} - \frac{d\theta_{pd}}{dx_a}\frac{d^2x_a}{ds^2}}{\frac{dx_a}{ds}^2}.$$

As $x_a$ is strictly increasing or decreasing according to $s$, $s$ can be uniquely determined for $x_a$. $\theta_{pd}$, $\frac{d\theta_{pd}}{dx_a}$ and $\frac{d^2\theta_{pd}}{dx_a^2}$ are obtained by table-lookup using this $s$.

The velocity and acceleration of the manipulator along the desired path are determined by the shape of the path and the initial velocity. Substituting $\ddot{e}_p = 0$ to Eq.(21), the acceleration $\ddot{x}_a$ of the active joints is,

$$\ddot{x}_a = -\frac{m_p^{-1}b + \frac{d^2\theta_{pd}}{dx_a^2}\dot{x}_a^2}{m_p^{-1}m_a + \frac{d\theta_{pd}}{dx_a}}. \quad (23)$$

$\frac{d\theta_{pd}}{dx_a}$ and $\frac{d^2\theta_{pd}}{dx_a^2}$ are functions of $x_a$. The angle $\theta_p$ of the passive joint is also a function of $x_a$, and then $m_a$ and $m_p$ are functions of $x_a$, too. Considering $\dot{\theta}_p = \frac{d\theta_{pd}}{dx_a}\dot{x}_a$ and Eq.(3)(13)(15), $b$ is a product of $\dot{x}_a^2$ and a function of $x_a$. Hence Eq.(23) can be rewritten as,

$$\ddot{x}_a = a(x_a)\dot{x}_a^2. \quad (24)$$

$a(x_a)$ is a function of $x_a$ determined by the shape of the path. If the time-axis is scaled by the factor $\kappa > 0$ as $s = \kappa t$,

$$\dot{x}_a = \kappa\frac{dx_a}{ds}, \quad \ddot{x}_a = \kappa^2\frac{d^2x_a}{ds^2}$$

then

$$\frac{d^2x_a}{ds^2} = a(x_a)(\frac{dx_a}{ds})^2 \quad (25)$$

Eq.(25) explicitly includes neither $t$ nor $\kappa$, and has the equivalent form of Eq.(24). Therefore, if the time-axis of the trajectory that satisfies Eq.(24) is scaled by a constant factor, it is also a solution of Eq.(24). It is the only solution for that scaling factor because of the uniqueness of the solution. Since the planned trajectory is one of the solutions of Eq.(24), all the trajectories when the manipulator is constrained to the desired path coincide with the trajectories to which the desired trajectory is time-scaled.

From

$$\ddot{x}_a = \frac{dx_a}{dt}\frac{d\dot{x}_a}{dx_a} = \frac{1}{2}\frac{d(\dot{x}_a)^2}{dx_a}, \quad (26)$$

Eq.(24) can be considered as a linear differential equation where $x_a$ is the independent variable and $\dot{x}_a^2$ is the unknown variable. The initial condition is given as $\dot{x}_a = \dot{x}_{a0}$ at $x_a = x_{a0}$, hence Eq.(24) can be solved as

$$\dot{x}_a^2(x_a) = \dot{x}_{a0}^2\exp(\int_{x_{a0}}^{x_a} 2a(z)dz). \quad (27)$$

$\exp(\int_{x_{a0}}^{x_a} 2a(z)dz)$ is a positive function of $x_a$, and equals 1 at $x_a = x_{a0}$. The time-scaling of the solution is obvious from Eq.(27), too.

### 4.3 Switchings of Control Law

When $e_p = 0$, $\dot{x}_a = 0$ and $\dot{\theta}_p = 0$, the active joint acceleration, $\ddot{x}_a$, calculated from Eq.(21) equals zero. This means the manipulator cannot start moving from the initial zero-velocity state by only the path tracking control. It is necessary to accelerate the manipulator with a feedforward control according to the desired trajectory of the active joint, $x_{ad}(t)$, $\dot{x}_{ad}(t)$ and $\ddot{x}_{ad}(t)$, at least in the initial part of Swing A.

The configuration where $m_p^{-1} m_a + \frac{d\theta_{pd}}{dx_a} = 0$ is singular in Eq.(21), and the feedback control in the previous section cannot be applied there. If $\dot{x}_a \to 0$ and $\dot{\theta}_p \to 0$, then $b(x_a, \theta_p, \dot{x}_a, \dot{\theta}_p) \to 0$ in Eq.(14) and $m_p^{-1} m_a + \frac{d\theta_p}{dx_a} \to 0$. Therefore, the initial and the desired configurations of the positioning are singular. Furthermore, $\frac{d\theta_{pd}}{dx_a}$ cannot be defined where $\frac{dx_a}{ds} = 0$ and $\frac{d\theta_{pd}}{ds} \neq 0$, and Eq.(18)(21) cannot be calculated. The end of Swing A and the start of Swing B come under this case because the active joints are at rest and the passive joint rotates there. The control law of Eq.(21) cannot be used in the neighborhood of these points.

The control law of the previous section does not depend on time and automatically scales the time-axis. It cannot be smoothly switched to the feedforward control that gives the active joint trajectory as a function of time. The active joint acceleration on the desired path is proportional to the square of the velocity, $\dot{x}_a^2$, from Eq.(24). We represent the desired trajectory of the active joints as functions of the position $x_a$, i.e. the desired acceleration is $\ddot{x}_{ad}(x_a)$ and the desired velocity is $\dot{x}_{ad}(x_a)$. The position where the control law is switched is $x_{a0}$ and the velocity at that time is $\dot{x}_{a0}$. When the active joint acceleration is given as

$$\ddot{x}_a = \frac{\dot{x}_{a0}^2}{\dot{x}_{ad}^2(x_{a0})} \ddot{x}_{ad}(x_a), \qquad (28)$$

Eq.(24) is satisfied at the moment of the switching and the control law is continuously changed. $\ddot{x}_{ad}(x_a)$ can be defined all over the trajectory and the control input can be always obtained. Since Eq.(28) has a form of a state feedback that does not depend on time, it is consistent with the time-scaling by the control law in the previous section.

## 5 Simulations and Experiments

We tested the proposed motion planning and feedback control method with the simulations and experiments using a two-axis planar manipulator (**Figure 3**) in which the first axis is active and the second axis is passive. $m$ is the mass of the second link, $I$ is the moment of inertia of the second link around the center of mass, $l$ is the distance between the passive joint and the center of mass of the second link, and $L$ is the length of the first link. The dynamic constraint due to the passive joint is,

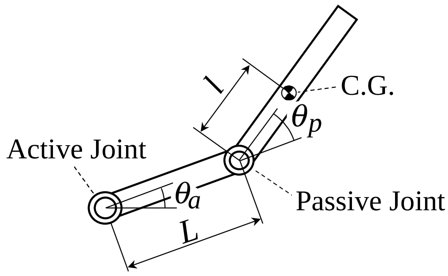


Fig. 3: Two-axis planar underactuated manipulator

$$(L\cos\theta_p + \lambda)\ddot{\theta}_a + \lambda\ddot{\theta}_p + L\sin\theta_p \dot{\theta}_a^2 = 0 \qquad (29)$$

where $\lambda \equiv l + I/(ml)$. $\lambda$ is the distance between the passive joint and the center of percussion of the second link.

First, the trajectories for the positioning are planned by the method of Section 3. The active joint trajectory of each swing, which corresponds to $\boldsymbol{f}(\kappa t)$ in Eq.(6), is given by

$$\begin{cases} \theta_a(t) &=& \theta_{a1} + (\theta_{a2} - \theta_{a1})(\kappa t - \frac{1}{2\pi}\sin 2\pi\kappa t) \\ \dot{\theta}_a(t) &=& \kappa(\theta_{a2} - \theta_{a1})(1 - \cos 2\pi\kappa t) \\ \ddot{\theta}_a(t) &=& 2\pi\kappa^2(\theta_{a2} - \theta_{a1})\sin 2\pi\kappa t \end{cases}$$

In Swing A, $\theta_{a1}$ is the initial angle $\theta_{a0}$, and $\theta_{a2}$ is the intermediate angle $\theta_{am}$. In Swing B, $\theta_{a1}$ is the intermediate angle $\theta_{am}$, and $\theta_{a2}$ is the desired angle $\theta_{ad}$. The maximum acceleration of the active joints is limited. The time-scaling factor, $\kappa_A$ and $\kappa_B$, for each swing is determined so that the peak angular acceleration, $2\pi\kappa^2(\theta_{a2} - \theta_{a1})$, for one swing equals the maximum and the acceleration for the other swing is within the limit.

The positioning trajectories are planned from the initial configuration $\theta_a = 0$[rad], $\theta_p = 0$[rad] to various desired configurations, and the intervals for the positionings are examined. The desired configurations are in the area of $0 \le \theta_a \le \pi$, $-\pi \le \theta_p \le \pi$ and the step angle is $\pi/12$. The intermediate angle of the active joint is searched within the initial angle $\pm\pi$ with $\pi/180$ step to find the angle where the positioning interval becomes minimum. The parameters of the manipulator are $L = 0.3$[m] and $\lambda = 0.2$[m]. The angular acceleration of the active joint is limited within

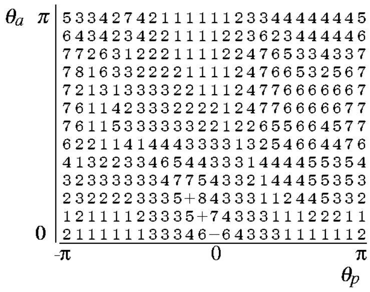


Fig. 4: Positioning intervals

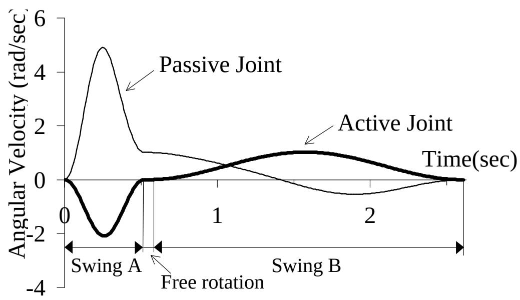


Fig. 5: Planned trajectory

$|\ddot{\theta}_a| \leq 4\pi$[rad/sec$^2$]. **Figure 4** shows the positioning intervals ([sec]) to the desired configurations. To the configurations marked as $+$, the positioning takes more than 10 seconds. The positioning trajectory cannot be found for the configuration marked as $-$. Positioning to most of the desired configurations can be achieved within 10 seconds.

**Figure 5** and **Figure 6** show an example of the planned trajectories. The initial configuration is $\theta_a =$ 0[rad], $\theta_p = 0$[rad] and the desired configuration is $\theta_a = 0.524$[rad], $\theta_p = 1.571$[rad]. The intermediate angle of the active joint is $\theta_a = -0.541$[rad]. The Swing A, the free rotation period, and Swing B take 0.520, 0.018 and 2.077 [sec], respectively. The total positioning interval is 2.615 seconds. Eq.(8) is numerically integrated using the fourth-order Runge-Kutta method. The trajectory for each swing is divided into 256 steps, and the intermediate angle is chosen from 40 candidates. The motion planning for one trajectory takes 8.8 [sec] with a personal computer (80486CPU, 50MHz).

Next, we verified the path tracking by the feedback control in Section 4. As the manipulator has one active joint, $x_a$ coincides with $\theta_a$. The desired path is the Swing B part of the positioning trajectory in **Figure 6**. An angle error of 0.1 [rad] is given to the passive joint at the start of Swing B. **Figure 7** shows the plot of the tracking error $e_p$. In the case of the feedforward control (thin line), the error increases and the manipulator cannot stop at the end of the trajectory. On the other hand, the feedback control suppresses the error and the manipulator follows the path (thick line). The manipulator stops at $\theta_a = 0.524$[rad], $\theta_p = 1.571$[rad] and positioning is achieved in spite of the initial error.

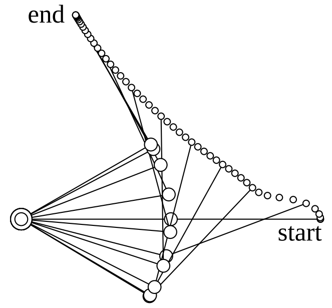


Fig. 6: Planned motion of manipulator

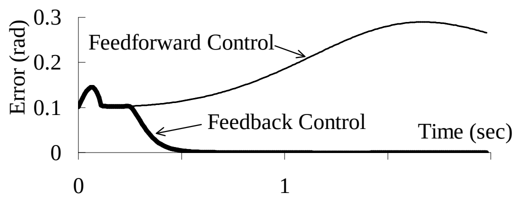


Fig. 7: Tracking error (simulation)

Then, the positioning of an experimental manipulator (**Figure 8**) is actually performed. The manipulator has two active joints and one passive joint. The base joint is fixed and the experiment is conducted using the remaining two joints. The length of each link is 0.3 m. The active joint is driven by a 20W dc motor with a reduction gear. The angle of each joint is measured by a rotary encoder. A personal computer (80486CPU, 50MHz) is used for the control. The feedback for the path tracking is applied during Swing B only. **Figure 9** shows the experimental result of the positioning from the initial configuration $\theta_a = 0.000$[rad], $\theta_p = 0.000$[rad] to the desired configuration $\theta_a = 0.524$[rad], $\theta_p = 1.571$[rad]. The manipulator stops at $\theta_a = 0.523$[rad], $\theta_p = 1.594$[rad] in 2.55 seconds. **Figure 10** also shows the positioning from $\theta_a = 0.523$[rad], $\theta_p = 1.474$[rad] to $\theta_a = 0$[rad], $\theta_p = 0$[rad]. The initial and the desired configurations are nearly swapped from the case of Figure 9. The ma-

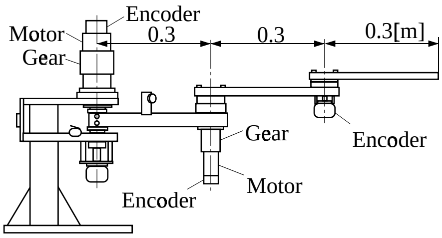


Fig. 8: Experimental setup

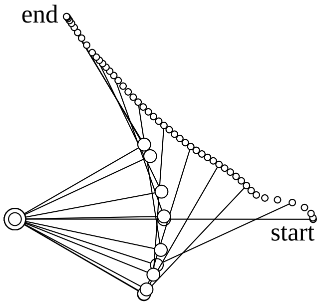


Fig. 9: Motion of manipulator (experiment)

nipulator reaches $\theta_a = -0.001$[rad], $\theta_p = -0.015$[rad] in 2.05 seconds.

## 6 Conclusions

We proposed a motion planning and feedback control method for the position control of an underactuated manipulator with one passive joint on which no gravity acts. The trajectory for positioning is planned by time-scaling of the trajectories that are numerically calculated from the initial and the desired configurations, and connecting them by the free rotation trajectory. This method gives the exact solution for the positioning, and the active and passive joints reach the desired angles simultaneously by only two swings of the active joints. Feedback control that makes the manipulator follow the desired path is also proposed. We experimentally demonstrated that position control to the desired configuration can be achieved by the proposed methods.

In future work, generalization of the proposed methods for general class of underactuated manipulators (more passive joints, with gravity and friction) should be discussed further. The robustness of the feedback control against the modelling error and the external disturbance should also be considered.

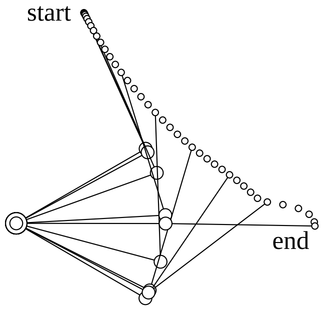


Fig. 10: Motion of manipulator 2 (experiment)

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