Abstract

The concept of discrete-time nonholonomic systems, in which the constraints are represented as difference equations of generalized coordinates, is introduced. Such systems can be seen in the digital control of continuous-time nonholonomic systems, and in mechanical systems with repetitive and discontinuous constraints. A two-wheeled mobile robot and pivoting manipulation of a polyhedral object are described as simple examples. The $k$-step reachable region is defined as the set of the $k$-th state which the system can reach from the initial state, and the reachability of such systems is discussed. A motion planning method using the Jacobian matrix of the state with regard to the input series is proposed.

1 Introduction

Studies on nonholonomic systems have attracted many researchers in the field of advanced robot control in recent years [1]. Nonholonomic systems in robotics involve kinematic or dynamic constraints represented as nonintegrable differential equations. The motion planning and control of such systems have been discussed in the framework of continuous-time nonlinear systems. Though the concept of nonholonomy originated in mechanics and has continuous properties inherently, this paper extends it into discrete-time systems. Discrete nonholonomic constraints formulated as difference equations are introduced based upon an analogy with continuous nonholonomic constraints.

This extension of nonholonomy allows the following two applications; digital control of continuous-time nonholonomic systems such as a wheeled vehicle and a space robot, and the control of mechanisms with repetitive, discontinuous constraints such as a legged robot and a multi-fingered hand. These systems are apparently in quite different classes, yet the concept of discrete-time nonholonomic systems can provide a unified theoretical basis for dealing with these systems.

The remainder of this paper is organized as follows. Section 2 introduces the concept of discrete-time nonholonomic systems. Section 3 presents two simple examples of discrete-time nonholonomic systems. Section 4 defines the $k$-step reachable region which the system can reach in $k$ steps, and discusses the reachability of the system. Section 5 presents the author’s proposed motion planning method for discrete-time nonholonomic systems.

2 Discrete-time Nonholonomic Systems

Before introducing discrete-time nonholonomic systems, a brief review of the definition of continuous-time nonholonomic systems is in order. A mechanical system which has a constraint that cannot be represented as $h(q,t) = 0$, ($q$: generalized coordinates, $t$: time) is called nonholonomic. Wheeled vehicles and space robots are typical examples of nonholonomic systems being studied in robotics. In those systems, the constraint can be formulated as a first-order differential equation,

$$h(q,\dot{q}) = 0$$

($q$: generalized coordinates, $\dot{q}$: generalized velocity)

The state equation is represented as a continuous-time nonlinear system,

$$\dot{q} = g(q,u) \quad (u: \text{input})$$

The number of inputs $u$ is generally smaller than the number of generalized coordinates $q$. In wheeled vehicles and space robots, the constraint (1) is expressed in a Pfaffian form, $H(q)\dot{q} = 0$, and the state equation (2) is a drift-free affine system, $\dot{q} = G(q)u$. 
Equations (1) and (2) are the expressions when the state variables change continuously over passage of time. These equations are the starting point from which nonholonomic systems have been investigated, mainly from the viewpoint of continuous-time nonlinear systems. Discrete-time nonholonomic systems can be formulated from an analogy with continuous-time systems. Let us suppose a system whose generalized coordinates change discretely with the discrete inputs. The generalized coordinates and the inputs at the k-th step are \( q^k = (q_1^k, ..., q_m^k)^T \in M \) and \( u^k = (u_1^k, ..., u_m^k)^T \in \Omega \), respectively, where \( m < n \).

**[Discrete-time Nonholonomic System]**

If the discrete-time system just described had a constraint that could not be represented as an algebraic equation,

\[
h(q^k, k) = 0
\]

it would be defined as a discrete-time nonholonomic system.

Similar to the continuous-time nonholonomic constraints as differential equations, there are discrete-time nonholonomic constraints as difference equations. A discrete-time nonlinear system,

\[
q^{k+1} = g(q^k, u^k)
\]

is a nonholonomic system if the constraint can be represented as \( n - m \) difference equations,

\[
h(q^{k+1}, q^k) = 0
\]

and cannot be transformed into Eq. (3). There might also be higher order nonholonomic systems in which the difference equation of the constraint includes \( q^{k+2}, q^{k+3}, ... \). This paper, however, focuses on first-order nonholonomic systems represented as Eq. (4) and (5).

Thus we have artificially extended the concept of nonholonomy to discrete-time systems, based on the analogy with continuous-time systems. The practical motivations for this extension include:

- Discrete-time control of a continuous-time nonholonomic system,
- Control of a mechanical system in which the constraint changes discontinuously and repetitively.

The first application is discretization and digital control of a continuous-time nonholonomic system. The control of a nonholonomic system usually involves complicated calculations that require computer control. In most cases, the controller is designed in the following two steps. First, an analogue controller is designed for the continuous-time model. Then, that controller is approximately discretized into a digital controller and applied to the actual system. It is preferable, however, to directly design a digital controller for the exactly discretized model. The effect of discretization errors could be taken into account particularly in the case of long sampling intervals.

In contrast to the vast amount of literature on continuous control of nonholonomic systems, there has not been much research reported on digital control. Nonetheless, several researchers have noted its importance. Divelbiss and Wen [2] proposed a nonholonomic motion planning method using a series of piecewise-constant inputs. Monaco, Normand-Cyrot, and their colleagues [3, 4, 5] intensively studied multirate digital control of nonholonomic systems. Their method is basically analogue-digital hybrid control. First, the original nonholonomic system is transformed into a chained form using a continuous state feedback. Then, piecewise-constant input is applied to the chained form with multirate sampling. Recently, Heimann [6] also proposed discrete-time open- and closed-loop control for an underactuated manipulator using piecewise constant torque inputs.

Here, a continuous-time nonholonomic system (2) is discretized with the sampling interval \( \Delta T \). State \( q^k \) of the discretized system is sampled from state \( q(t) \) of the original continuous system as \( q^k = q(k \Delta T) \) \( (k = 0, 1, ...) \). Piecewise constant inputs, \( u(t) = u^k \), are applied to the continuous system during each interval \( k \Delta T < t < (k + 1) \Delta T \). Then the state at the next step is,

\[
q^{k+1} = q^k + \int_{k \Delta T}^{(k+1) \Delta T} g(q, u^k) \, dt
\]

Since \( u^k \) and \( \Delta T \) are constant, the integral results in a function of \( q^k \) and \( u^k \). Eq. (6) now has the same form as Eq. (4). It should be noted that input \( u(t) \) does not have to be piecewise-constant. It can be a continuous function parameterized by \( u^k \), e.g., quasi-periodical input of interval \( \Delta T \).

Another situation for a discrete-time nonholonomic system appears when the constraint changes repetitively and discontinuously. Transition of the state inherently occurs in a discrete way. Such a system can be seen in a legged robot or a multi-fingered robot hand. The constraint discontinuously changes according to the contact status between the foot and the ground or between the fingertip and the manipulated object. The robot body or the object is driven by the repetitive movement of the legs or fingers. The motion
variables of each leg or finger (e.g. step distance) are the discrete inputs, and at the end of each cycle the configurations of the robot body and the legs, or the object and the fingers, are the state variables.

Kelly and Murray [7] and Goodwine and Burdick [8] investigated the nonholonomy of a legged robot. Their studies treated the whole system as a collection of continuous systems connected by the change of the constraint. In contrast, the author’s proposed method describes the motion of the system during each constraint with a few variables and treats them as discrete inputs. Thus, the problem is simplified to the discrete transition of the state.

3 Examples of Discrete-time Nonholonomic Systems

Two simple examples of discrete-time nonholonomic systems are presented here. One example demonstrates the discretization of a continuous-time nonholonomic system, and the other, a mechanism with repetitive and discontinuous constraint. Common characteristics of discrete-time nonholonomic systems can be seen in these examples.

3.1 Discretization of a continuous-time nonholonomic system

Discretization of a two-wheeled mobile robot (Fig. 1), a classical example of a continuous nonholonomic system, is considered here. Inputs, \( \omega_L \) and \( \omega_R \), are the angular velocity of the left and the right wheels. The state equation of this continuous system is,

\[
\begin{align*}
\dot{x} &= \frac{d(\omega_R + \omega_L)}{2} \cos \theta \frac{1}{\Delta T} \\
\dot{y} &= \frac{d(\omega_R + \omega_L)}{2} \sin \theta \frac{1}{\Delta T} \\
\dot{\theta} &= \frac{d(\omega_R - \omega_L)}{l} \frac{1}{\Delta T}
\end{align*}
\]  

Inputs, \( \omega_R \) and \( \omega_L \), are given piecewise-constant, \( \omega_R^k \) and \( \omega_L^k \), with sampling interval \( \Delta T \). The discrete inputs are,

\[
\begin{align*}
u_v^k &= d\Delta T(\omega_R^k + \omega_L^k)/2, \\
u_\omega^k &= d\Delta T(\omega_R^k - \omega_L^k)/l
\end{align*}
\]

Then, the state equation of the discretized system is,

\[
\begin{align*}
x^{k+1} &= x^k + \{ \sin(\theta^k + u_v^k) - \sin(\theta^k) \} u_v^k / u_v^k \\
y^{k+1} &= y^k - \{ \cos(\theta^k + u_\omega^k) - \cos(\theta^k) \} u_\omega^k / u_\omega^k \\
\theta^{k+1} &= \theta^k + u_\omega^k 
\end{align*}
\]

(When \( u_\omega^k \neq 0 \))

\[
\begin{align*}
x^{k+1} &= x^k + \cos(\theta^{k+1} - \cos(\theta^k)) + \\
y^{k+1} &= y^k + \sin(\theta^{k+1} - \sin(\theta^k)) \\
\theta^{k+1} &= \theta^k 
\end{align*}
\]

(When \( u_\omega^k = 0 \))

When inputs \( u_v^k \) and \( u_\omega^k \) are removed from Eq. (8), the constraint of the system is obtained as a difference equation,

\[
(x^{k+1} - x^k)(\cos(\theta^{k+1} - \cos(\theta^k)) + \\
y^{k+1} - y^k)(\sin(\theta^{k+1} - \sin(\theta^k)) = 0
\]

Since Eq. (9) also satisfies Eq. (10), the constraint is described by this equation for both cases.

Eq. (8) and (9) can be linearized around the origin with regard to the small inputs as follows.

\[
q^{k+1} = \left( \begin{array}{ccc} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{array} \right) q^k + \left( \begin{array}{c} 0 \\
0 \\
0 \end{array} \right) u^k
\]

\[
(q^k = (x^k, y^k, \theta^k)^T, u^k = (u_v^k, u_\omega^k)^T)
\]

This system does not satisfy the controllability condition of a discrete-time linear system. Hence, the first linear approximation (11) of the system (8), (9) is uncontrollable.

Then, let us suppose the following series of control inputs are applied to the nonlinear system (8) and (9),

\[
\begin{align*}
\bar{u}_v^0 &= (\delta, 0)^T \\
\bar{u}_v^1 &= (\delta, \pi)^T, \bar{u}_v^1 &= (\delta, -\pi)^T \\
\bar{u}_\omega^0 &= (0, \delta)^T \\
\bar{u}_\omega^1 &= (0, 0, \delta)^T
\end{align*}
\]

where \( \bar{q}_v^0 = (0, 0, 0)^T \). The state can transition to all three directions of \( x, y, \theta \) using the two inputs, \( u_v^k \) and \( u_\omega^k \), as is the case with the continuous-time nonholonomic system. Also from this fact, the constraint cannot be represented as the form of Eq. (3).
3.2 Repetitive and discontinuous constraint

Aiyama et al. [9] proposed a pivoting manipulation method for manipulating a polyhedral object on a plane without picking it up. The object is inclined and supported on a vertex, and then it is rotated around this point (Fig. 2). Repeating this motion around the two vertices of the bottom edge alternately, the object “walks” on the plane and its position and orientation can be controlled.

Figure 3 shows the kinematic model of the pivoting. Edge AB is first rotated around vertex B by the angle $\theta^k_B$, and is next rotated around A by $\theta^k_A$. The sequence of these two rotations is considered to be one step of the transition. The state equation is,

$$
\begin{align*}
{x^{k+1}} &= x^k + l \cos \theta^k - l \cos(u^k_A - u^k_B) \\
{y^{k+1}} &= y^k + l \sin \theta^k + l \sin(u^k_A - u^k_B) \\
{\theta^{k+1}} &= \theta^k + u^k_A - u^k_B
\end{align*}
$$

and it has the same form as Eq. (4). The biped robot with a constant step [10] illustrated in Fig. 4 has equivalent kinematics.

When inputs $u^k_A$ and $u^k_B$ are removed from Eq. (12), the constraint of the system is obtained as a difference equation,

$$(x^{k+1} - x^k)^2 + (y^{k+1} - y^k)^2 = l^2$$

This equation also has the same form as Eq. (5).

Equation (12) can be linearly approximated around the origin as follows.

$$
q^{k+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} q^k + \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} u^k
$$

$$(q^k = (x^k, y^k, \theta^k)^T, \; u^k = (u^k_A, u^k_B)^T)$$

This system is also uncontrollable.

Each of the following input series leads, respectively, to the motion of only one coordinate (Fig. 5).

- $u^0 = (\delta, \delta)^T$, $u^1 = (-\delta, -\delta)^T$ \rightarrow $q^2 = (2l(1 - \cos \delta), 0, 0)^T$
- $u^0 = (\delta, \delta)^T$, $u^1 = (0, \delta)^T$ \rightarrow $q^2 = (0, 2l \sin \delta, 0)^T$
- $u^0 = (\delta, 0)$ \rightarrow $q^1 = (0, 0, \delta)^T$

where, $q^0 = (0, 0, 0)^T$. The system can move in any direction, though its linear approximation is uncontrollable.

4 k-Step Reachable Region

This section discusses the properties of the reachable region of a discrete-time nonholonomic system, i.e., where the system can reach from an initial state. For a continuous-time system, the inputs applied over a period of time can be considered an input series of infinite dimensions. The number of inputs to a discrete-time system over a given period of time, however, is
finite. This results in the different characteristics of the reachable region of a continuous-time system and a discrete-time system.

In the discrete-time system (4), the input series and the state is related as,

\[
\begin{align*}
q^i &= g(q^0, u^i) = G_i(q^0, u^0) \\
q^2 &= g(q^1, u^i) = G_2(q^1, u^1) \\
&\vdots \\
q^k &= g(q^{k-1}, u^{k-1}) = G_k(q^0, u^0, \ldots, u^{k-1}) 
\end{align*}
\]  

(15)

If \(u^i\) and \(u^j\) (\(i \neq j\)) are considered to be individual inputs, the number of inputs that affect the final state increases every step. Therefore, the properties of the reachable region depend on the number of steps the system passes through. Now, the reachable region related to the number of steps is considered. A \(k\)-step reachable region \(\Lambda(q^0, k)\) from an initial state \(q^0\) is defined as the set of the \(k\)-th state \(q^k\) reachable from \(q^0\) with admissible inputs.

\[
\Lambda(q^0, k) = \{q^j; q^{j+1} = g(q^j, u^j), u^j \in \mathcal{O}, 0 \leq j \leq k - 1\} 
\]  

(16)

Figure 6 shows 1-step and 2-step reachable regions from the origin \((0,0,0)^T\) in the pivoting manipulation described in Section 3.2. The 1-step reachable region is limited to the surface of a cylinder with radius \(l\) (Fig. 6(a)). The three generalized coordinates cannot be transferred to the desired value simultaneously in one step. The 2-step reachable region is the surface and the inside of a cylinder with radius \(3\) (Fig. 6(b)). The state can reach any point within this region in two steps. Thus, the dimension of the reachable region of a discrete-time nonholonomic system depends on the number of steps.

Next, \(k\)-step local reachability is defined, following the definition of local reachability of a continuous-time system. If the \(k\)-step reachable region \(\Lambda(q^0, k)\) from initial state \(q^0\) has an interior within the state space \(M\), this system is \(k\)-step locally reachable from \(q^0\).

Eq. (15), \(q^k = G_k(q^0, u^k, u^1, \ldots, u^{k-1})\), means that the \(k\)-th state is determined as a nonlinear function of input series \(U_k = (u^1, \ldots, u^{k-1})^T\) comprising \(m \times k\) components. The Jacobian matrix, i.e., the partial derivative of \(G_k\) with regard to \(U_k\),

\[
J_k = \frac{\partial G_k}{\partial U_k} = \left(\frac{\partial G_k}{\partial u^0}, \ldots, \frac{\partial G_k}{\partial u^{k-1}}\right) 
\]  

(17)

is considered. If the input series is perturbed as \(U_k = U_{k0} + \delta U_k,\) the state reaches \(G_k(q^0, U_{k0}) + \delta q^k\), where,

\[
\delta q^i = J_k \delta U_k + O(|\delta U_k|^2) 
\]

If the rank of \(J_k\) equals the dimension of state space \(M\), it is obvious that the state can move in any direction of \(M\), by adding a small perturbation to input series \(U_k\). If there is an input series \(U_k\) such that

\[
\text{rank } J_k = \dim M 
\]

(18)

then the \(k\)-step reachable region \(\Lambda(q^0, k)\) has an interior in \(M\) and the system is \(k\)-step locally reachable. Jakubczyk and Sontag [11] defined the forward/backward accessibility of discrete-time nonlinear systems. They showed Eq. (18) and equivalent conditions using a vector field generated from the discrete-time state equation. Though Eq. (18) is a simple condition, Jacobian matrix \(J_k\) relates the modification of the input series to the change of the resulted state, and this can be useful in motion planning and control.

The Jacobian matrix \(J_k\) of a discrete-time linear system \(q^{k+1} = Aq^k + Bu^k\) becomes \(J_k = (A^{k-1}B, \ldots, AB, B)\). This is equivalent to the controllability matrix when \(k = n\). If the linear system is controllable, it is also \(n\)-step locally reachable. The controllability index equals the minimum step \(k\) that satisfies Eq. (18). Hence, condition (18) is a natural extension of the controllability condition of linear systems.

Next, the step local reachability of pivoting manipulation is examined. For Jacobian matrix \(J_2\) with regard to 2-step pivoting from origin \((0,0,0)^T\),

\[
|J_2J_2^T| = l^4\left(3 - \cos u_A^0 - \cos(2u_A^1) + \cos u_B^0 + 2u_B^0 - \cos(2u_A^1 - u_B^0) + \cos(2u_A^1 - u_B^0) - \cos u_B^0\right)
\]

Some input series, e.g., \(u_A^0 = 0, u_B^0 = 0\), makes the preceding equation zero. However, the rank of \(J_2\) is three otherwise, and the system is thus 2-step locally reachable.
5 Motion Planning

This section describes the author’s proposed motion planning method for discrete-time nonholonomic systems. The planning problem implies the determination of the input series \( U_k = (u^0, \ldots, u^{k-1}) \) that transfers the system \( (4) \) from initial state \( q^0 \) to the desired state \( q^k \) in \( k \) steps. From Eq. (15), the planned input \( U_k \) is a solution of the nonlinear algebraic equation \( q^k = G_k(q^0, U_k) \).

This equation has the same form as a kinematic equation of a redundant manipulator if the input series \( U_k \) and the desired state \( q^k \) are replaced with the joint angles and the operational coordinates, respectively. Therefore, motion planning of a discrete-time nonholonomic system has a structure common to the inverse kinematics problem of a manipulator. The motion planning problem can be solved by a similar method with the resolved rate motion control of a redundant manipulator \([12]\), using the Jacobian matrix, \( J_k \). From the viewpoint of numerical calculation, it can also be interpreted as a type of Newton-Raphson method.

First, the pseudo-inverse of the Jacobian matrix, \( J_k^+ = J_k^T (J_k J_k^T)^{-1} \), is calculated for some initial value of the input series. Then, the perturbation of the input series is calculated.

\[
\Delta U_k = J_k^+ K_p (q^k - q^0) + (I - J_k^+ J_k) f(U_k) \tag{19}
\]

The first term reduces the error between the \( k \)-th state and the desired state. \( K_p > 0 \) is the error feedback gain. \( f(U_k) \) in the second term is a subtask vector utilizing the redundancy of the inputs. It can perform lower priority tasks such as input minimization and obstacle avoidance. Due to the null space matrix, this term does not affect the \( k \)-th state.

The perturbation (19) is added to the input series, \( U_k \leftarrow U_k + \Delta U_k \), and the \( k \)-th state is then evaluated again for the new input. These calculations are repeated until the error \( q^k - q^0 \) decreases sufficiently. Then the input series \( U_k \) is obtained (Fig. 7).

The calculation of the Jacobian matrix, \( J_k \), seems to require a symbolic computation of the composite function \( G_k(q^0, U_k) \) and its partial derivative. However, the submatrices composing \( J_k \) can be simplified using the one-step transition function \( g(q, u) \) as follows.

\[
\frac{\partial G_k}{\partial u} = \frac{\partial g}{\partial q} \bigg|_{q^{-1}} \cdots \frac{\partial g}{\partial q} \bigg|_{q^{k+1}} \frac{\partial g}{\partial u} \bigg|_{q^k}
\]

Therefore, complicated symbolic operations are not necessary.

Applying the above method, the motions of the wheeled mobile robot in Section 3.1 are planned. The subtask using the redundant inputs is given as \( f(U_k) = - (K_v u_v^0, K_w u_w^0, \ldots, K_v u_v^{k-1}, K_w u_w^{k-1}) \). This minimizes the cost function, \( \sum_{i=1}^{k} \{K_v (u_v^i)^2 + K_w (u_w^i)^2 \} \). The planned motions are illustrated in Fig. 8. The error of the \( k \)-th state from the desired state converges to zero, Fig. 9 (a), and the cost function reduces, (b), while iteratively calculating the trajectory Fig. 8 (a).

The pivoting motions in Section 3.2 are also planned (Fig. 10). The subtask is \( f(U_k) = - (u_A^0, u_B^0, \ldots, u_A^{k-1}, u_B^{k-1}) \) and the cost function is \( \sum_{i=1}^{k-1} \{ (u_A^i)^2 + (u_B^i)^2 \} \). The computer programs for the planning of the mobile robot and pivoting manipulation are almost the same except for the calculations of the transition functions and their partial derivatives. Motion planning is achieved by an unified method in both cases.
6 Conclusions

The concept of nonholonomy was extended to discrete-time systems through an analogy with continuous-time systems. Potential applications of this extension include digital control of continuous-time nonholonomic systems, and control of mechanisms with repetitive and discontinuous constraints. The digital control of a two-wheeled vehicle, and pivoting manipulation of a polyhedral object were shown as examples. In both cases, the constraints are represented as difference equations, and the states can move in any direction in spite of the uncontrollability of the linear approximations. The dimension of the region the state can reach depends on the number of steps. The input series to reach the desired state can be solved in the same way as the inverse kinematics of a redundant manipulator.

References


