

# **A sequent calculus for Limit Computable Mathematics**

Stefano Berardi and Yoriyuki Yamagata

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- Part of Hayashi's "Proof Animation Project"

# LCM and classical logic

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$$EM_1(P) \equiv \forall x(\exists y Pxy \vee \forall y \neg Pxy)$$

$P$  : decidable, is valid in LCM, while

$$EM_2(Q) \equiv \forall x(\exists y \forall z Qxyz \vee \forall y \exists z \neg Qxyz)$$

$Q$  : decidable, is **not** valid.

# Strength of LCM

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- Conjecture : Intuitionism +  $EM_1$

# Game semantics of LCM

**1-bck. game** : Simple extension of Lorenzen/Hintikka game

**Theorem.** (Berardi, Coquand, Hayashi 2005)

$A$  is valid in LCM  $\Leftrightarrow$  Prover ( $\mathcal{E}$ ) is winning in 1-bck. game of  $A$ .

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## Isomorphism Theorem.

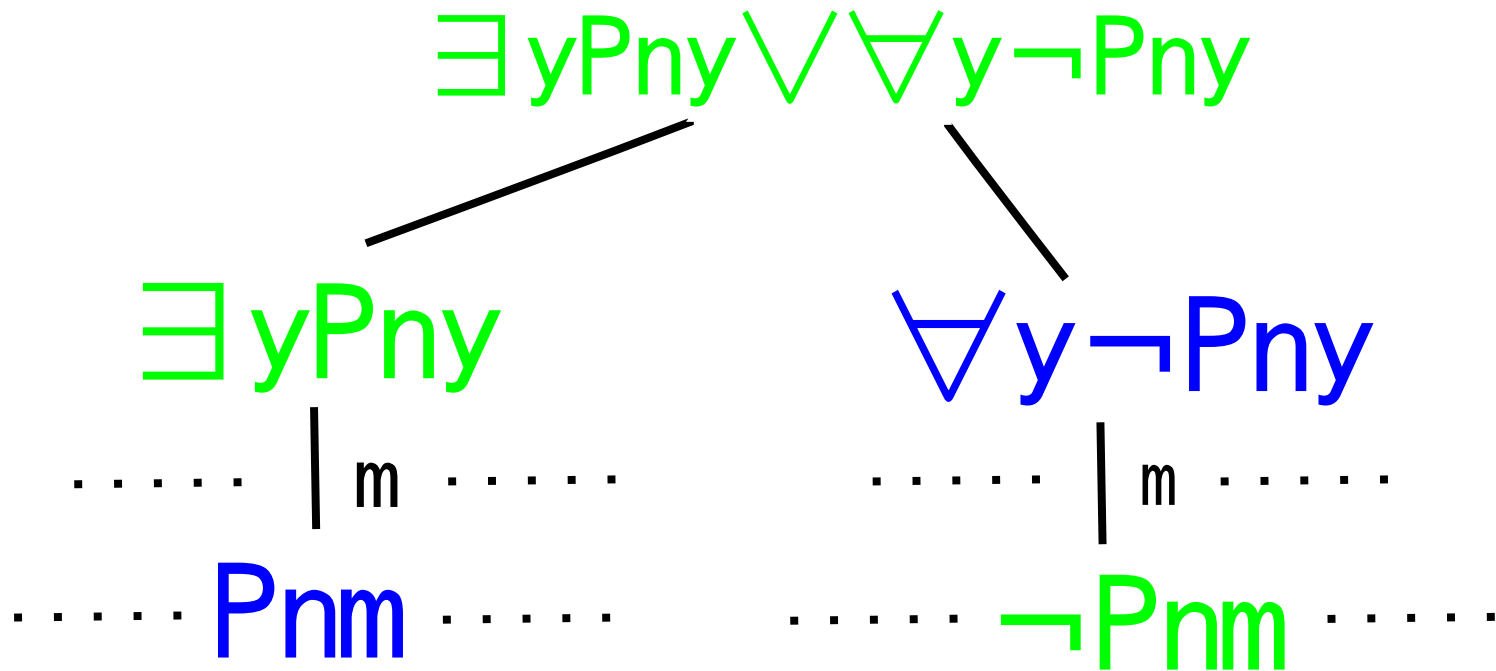
A proof  $\pi$  of formula  $A$  in  $PA_1$

$\leftrightarrow$  1:1, tree-iso.

a winning strategy of 1-bck. game of  $A$ .

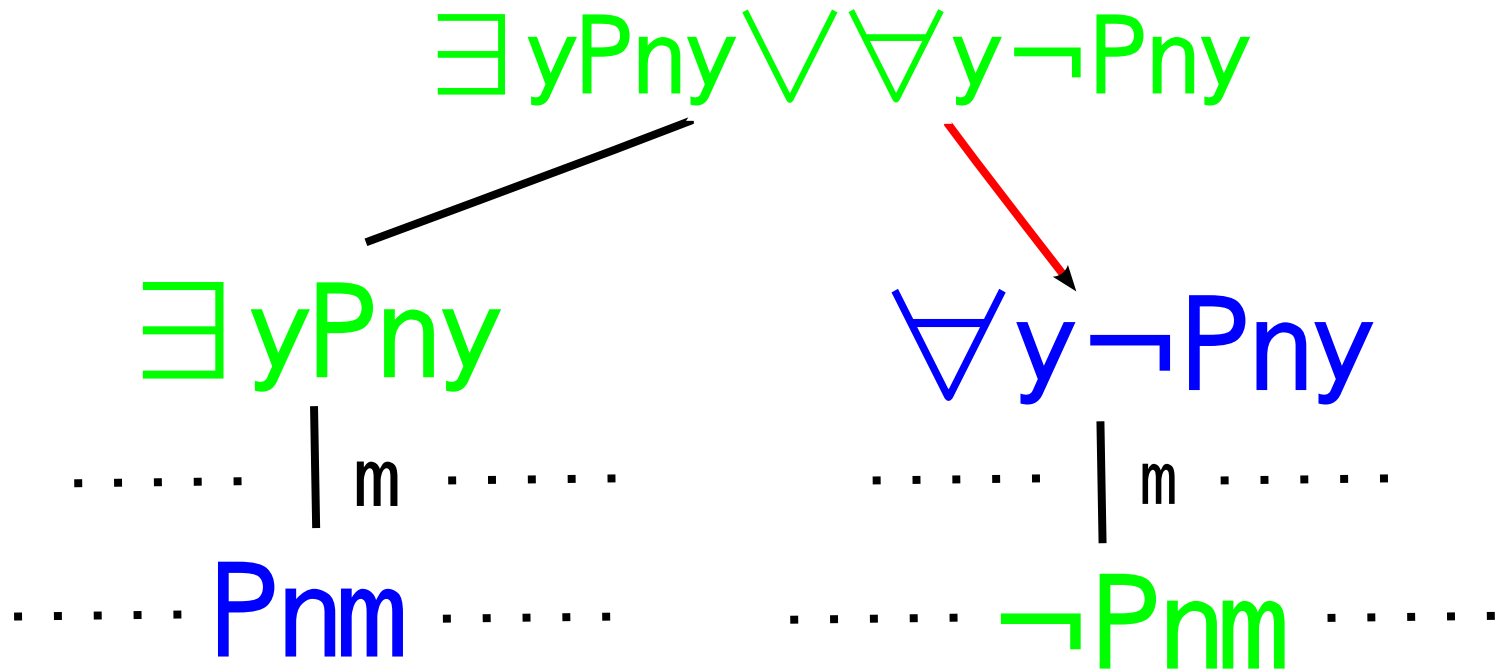
# Lorenzen/Hintikka game

2-person game between  $\mathcal{E}$  and  $\mathcal{A}$ . Conjunctions and false atomics are played by  $\mathcal{E}$ , otherwise positions are played by  $\mathcal{A}$ .  $Pnm$  below is true.



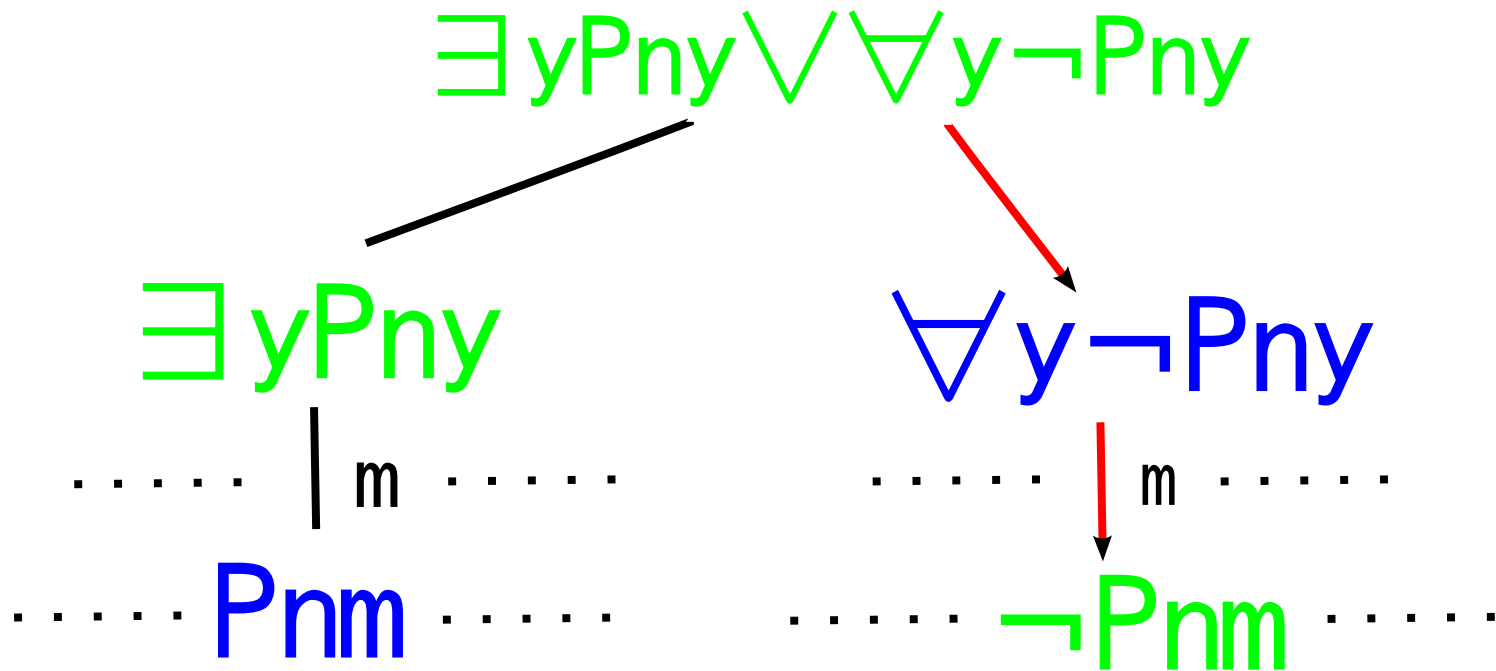
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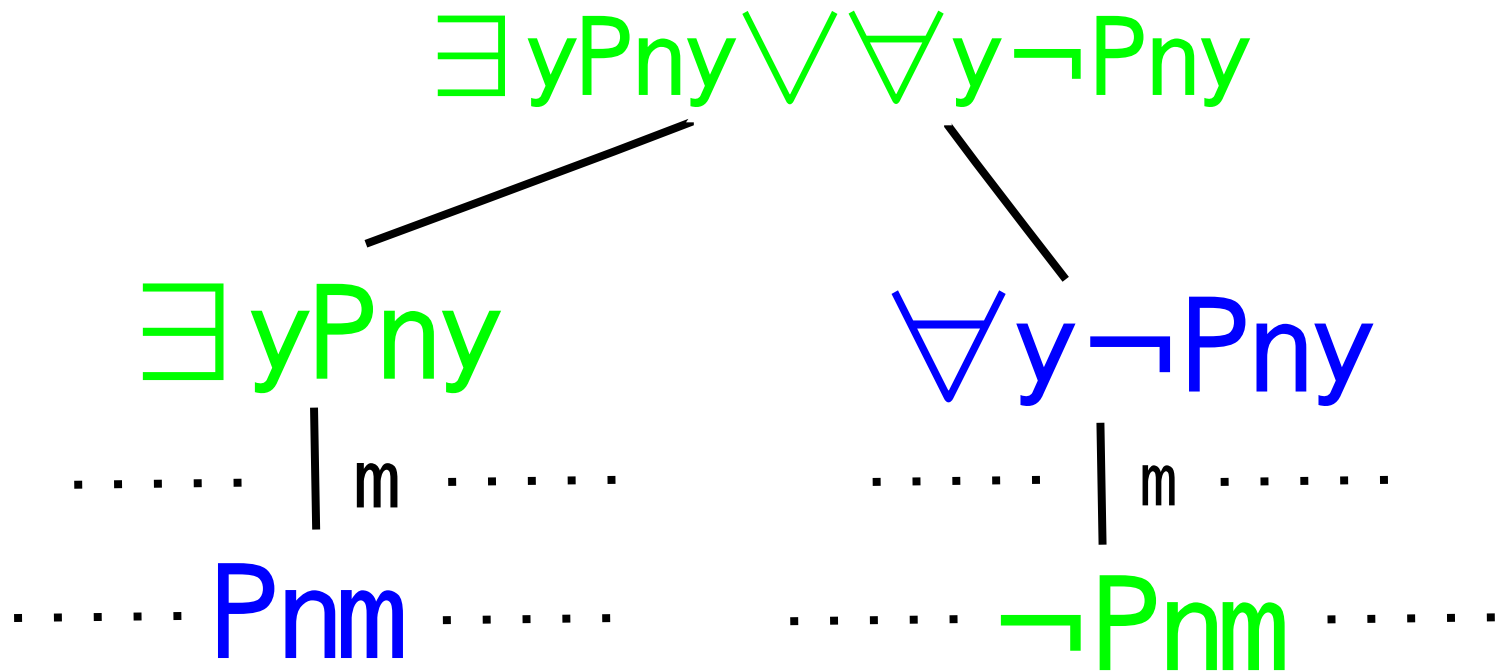
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$\mathcal{E}$  loses here.

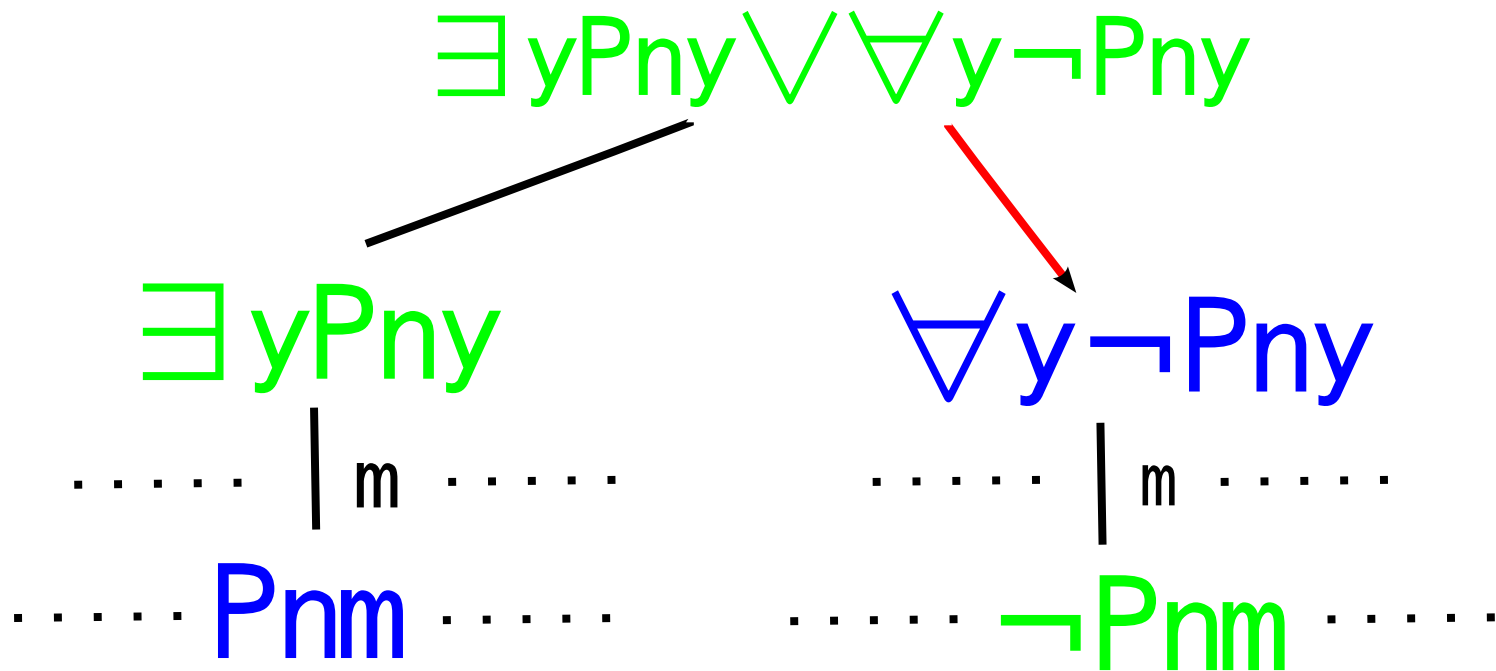
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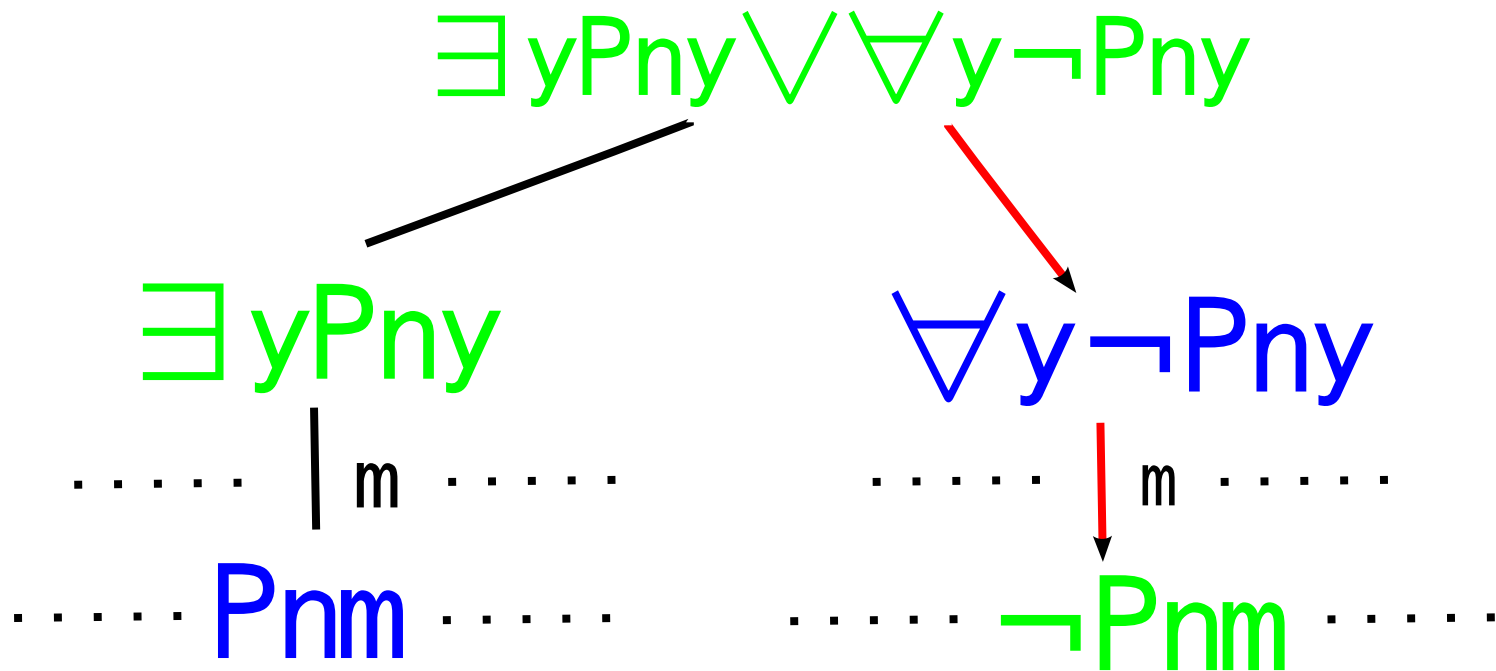
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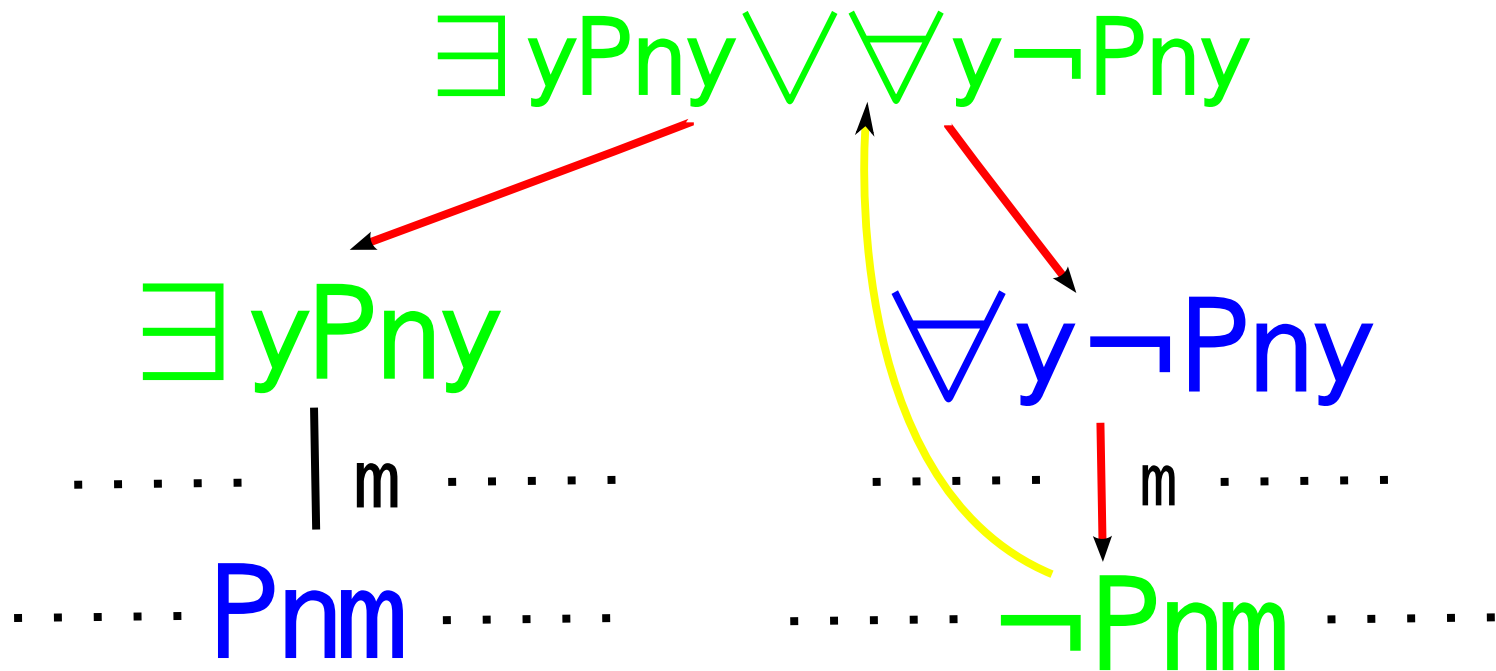
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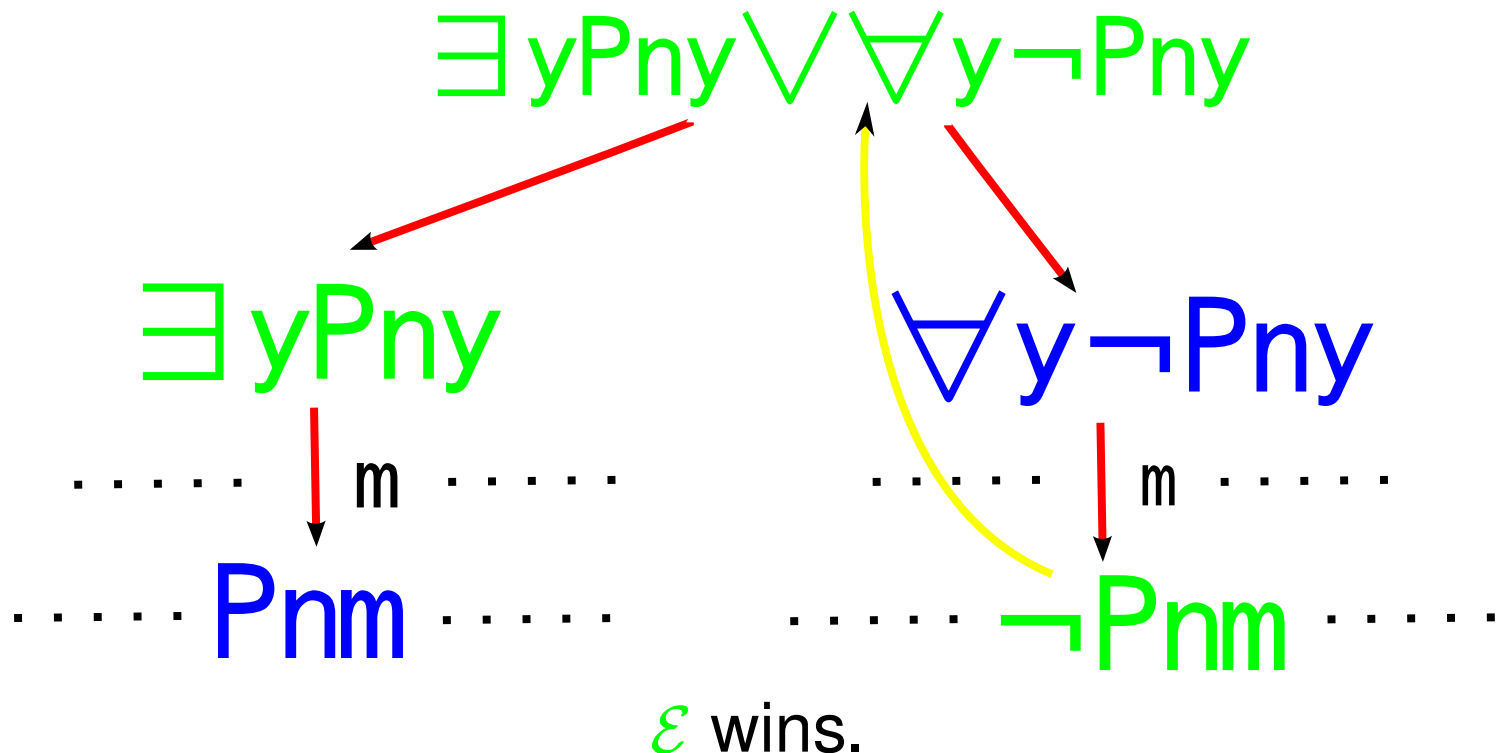
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Formulas  $F ::= Px \mid F \wedge F \mid F \vee F \mid \forall x F \mid \exists x F$

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This interpretation naturally leads to inference rules.

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Since  $\mathcal{E}$  is going to win in this position, no more need of strategies.

# $PA_1$ : Conjunctions

$\mathcal{A}$  moves at conjunctions

$$\frac{\vdash \Gamma, A_1 \quad \vdash \Gamma, A_2}{\vdash \Gamma, A_1 \wedge A_2} \wedge$$

$$\frac{\vdash \Gamma, A(0) \quad \dots \quad \vdash \Gamma, A(n) \quad \dots}{\vdash \Gamma, \forall x A(x)} \forall$$

$\mathcal{E}$  prepares all possible moves of  $\mathcal{A}$ .

# $PA_1$ : Disjunction

$$\frac{\vdash \Gamma, A_1 \vee A_2, A_i}{\vdash \Gamma, A_1 \vee A_2, \Delta} \vee$$

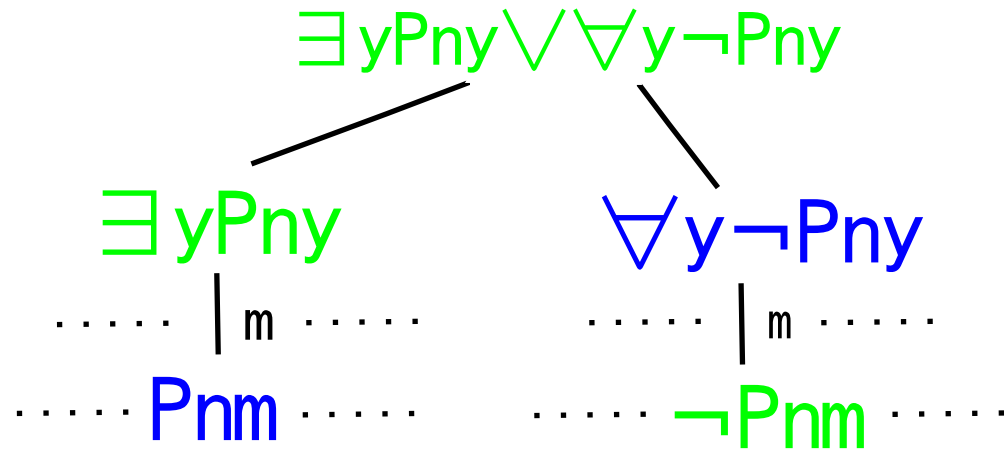
$\mathcal{E}$  retracts all moves in  $\Delta$  and backtracks to  $A_1 \vee A_2$ , then chooses a node  $A_i$ .

# $PA_1$ : Disjunction

$$\frac{\vdash \Gamma, \exists x A(x), A(n)}{\vdash \Gamma, \exists x A(x), \Delta} \exists$$

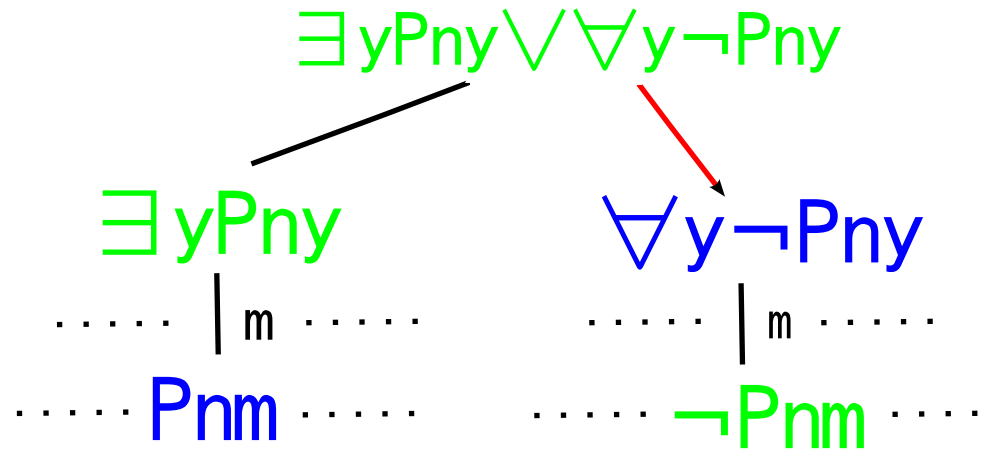
$\mathcal{E}$  retracts all moves in  $\Delta$  and backtracks to  $\exists x A(x)$ , then chooses a node  $A(n)$ .

# Proof and Game play



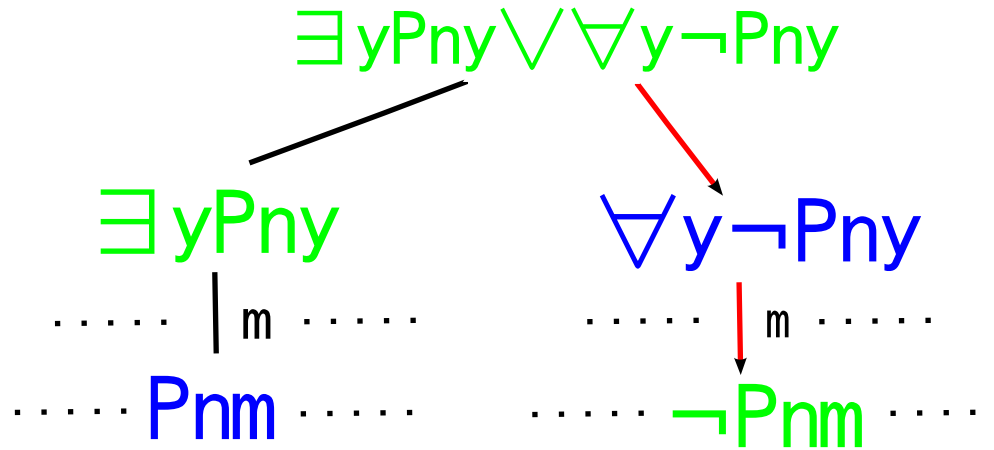
$$\vdash \exists y Pny \vee \forall y \neg Pny$$

# Proof and Game play



$$\frac{\vdash \exists y Pny \vee \forall y \neg Pny, \forall y \neg Pny}{\vdash \exists y Pny \vee \forall y \neg Pny} \vee$$

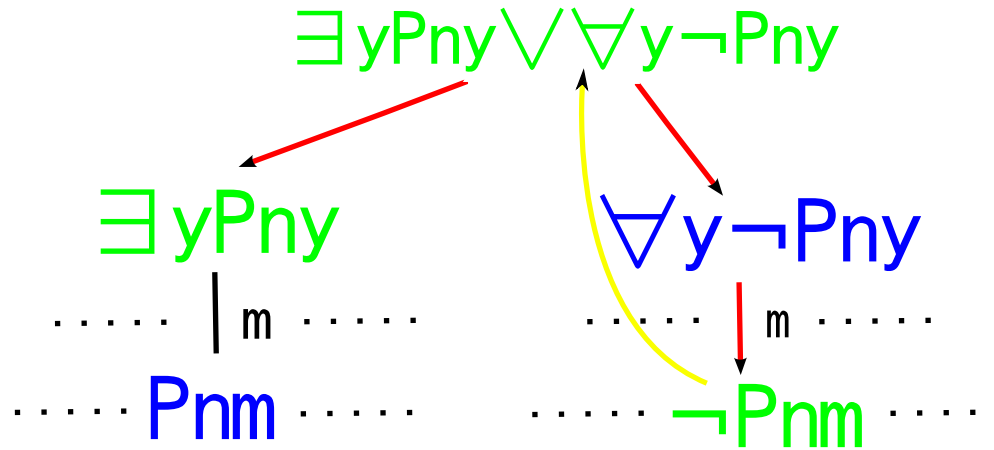
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# Conclusion

- We introduce a proof system  $PA_1$ , an  $\omega$ -logic without Exchange
- We show proofs of formula  $A$  in  $PA_1$  and winning strategies of 1-bck. games over  $A$  has a tree-isomorphism

# Future work

- Interpretation of Cut-rule.
- Interpretation of implication and Modus ponens
- Relation to cut-elimination

The End