Sequent calculus for 2-backtracking

Preliminary work

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Agenda

1. 1-backtracking game
2. 2-backtracking game
3. CL(2) - current attempt

Part 1

1-backtracking game

Game

- G : a two person game between Eloise and Abelard.
- $G \equiv \langle M, \curvearrowleft, t \rangle$ where
  - M : set of moves
  - $\curvearrowleft \subset M \times M$: "answering" relation
  - $t : M \rightarrow \{\text{Eloise}, \text{Abelard}\}$: turn map

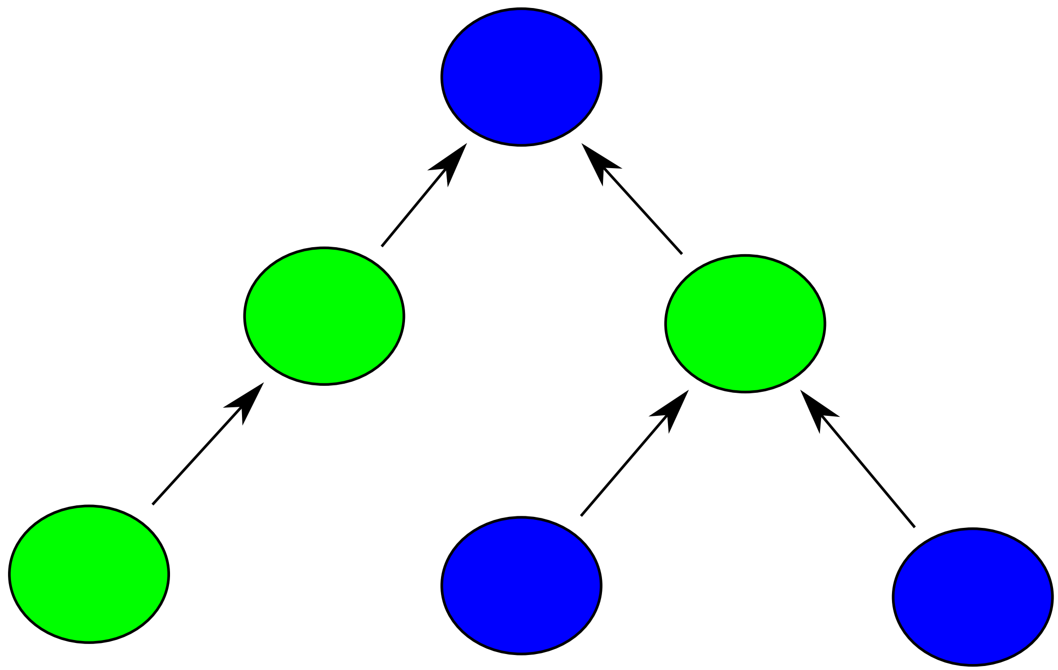
  If $t(m) =$ Eloise, we call m a E-move.
  If $t(m) =$ Abelard, we call m a A-move.

1-backtracking game G^1 of G

- A game building a *position tree* by Eloise and Abelard.
- What is a position tree?
  - tree built by moves of G.
  - each child "answers" its parent.
- Turn is determined by the rightmost-undermost element.

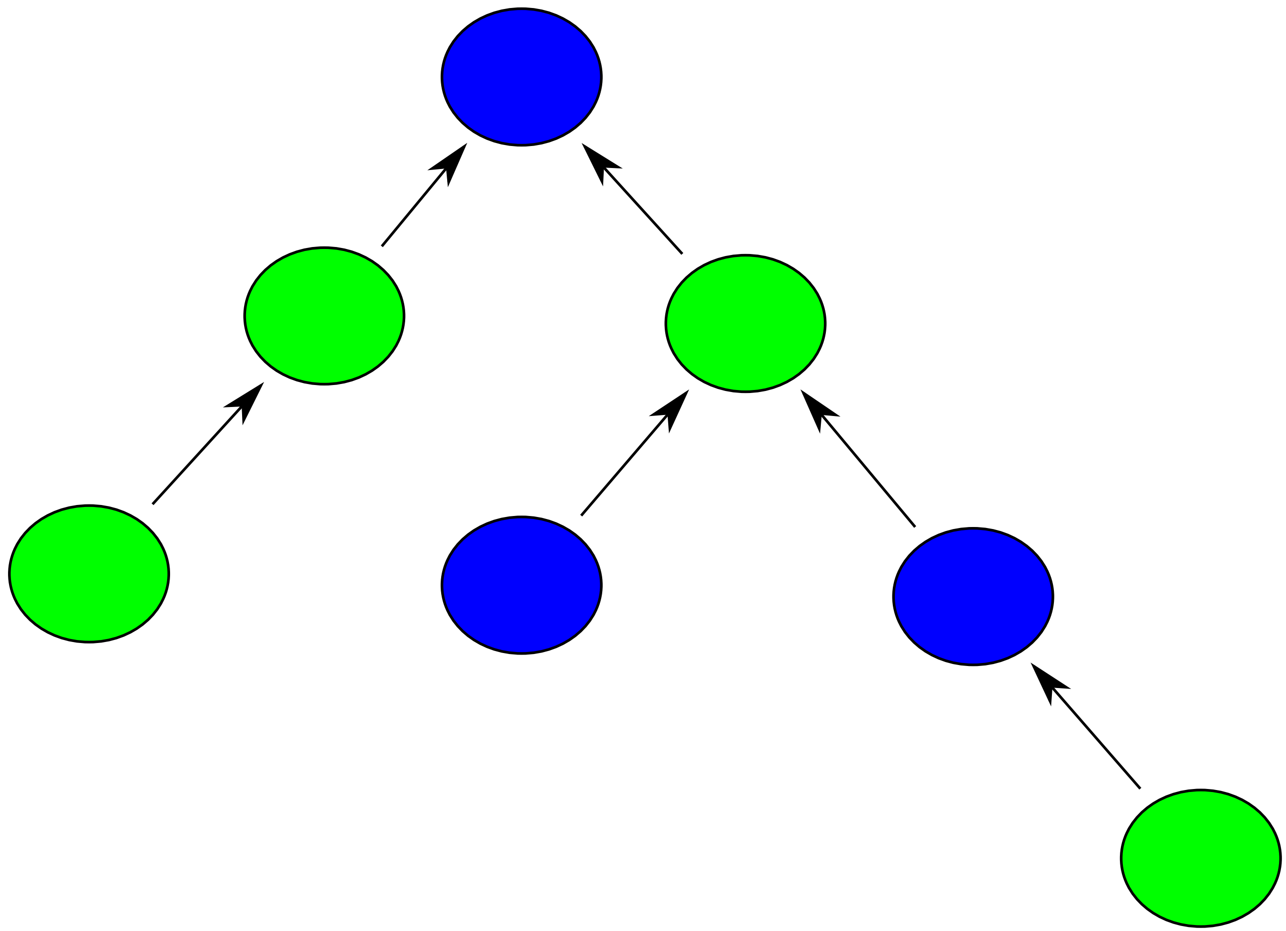
1-backtracking game G^1 of G

Abelard's turn



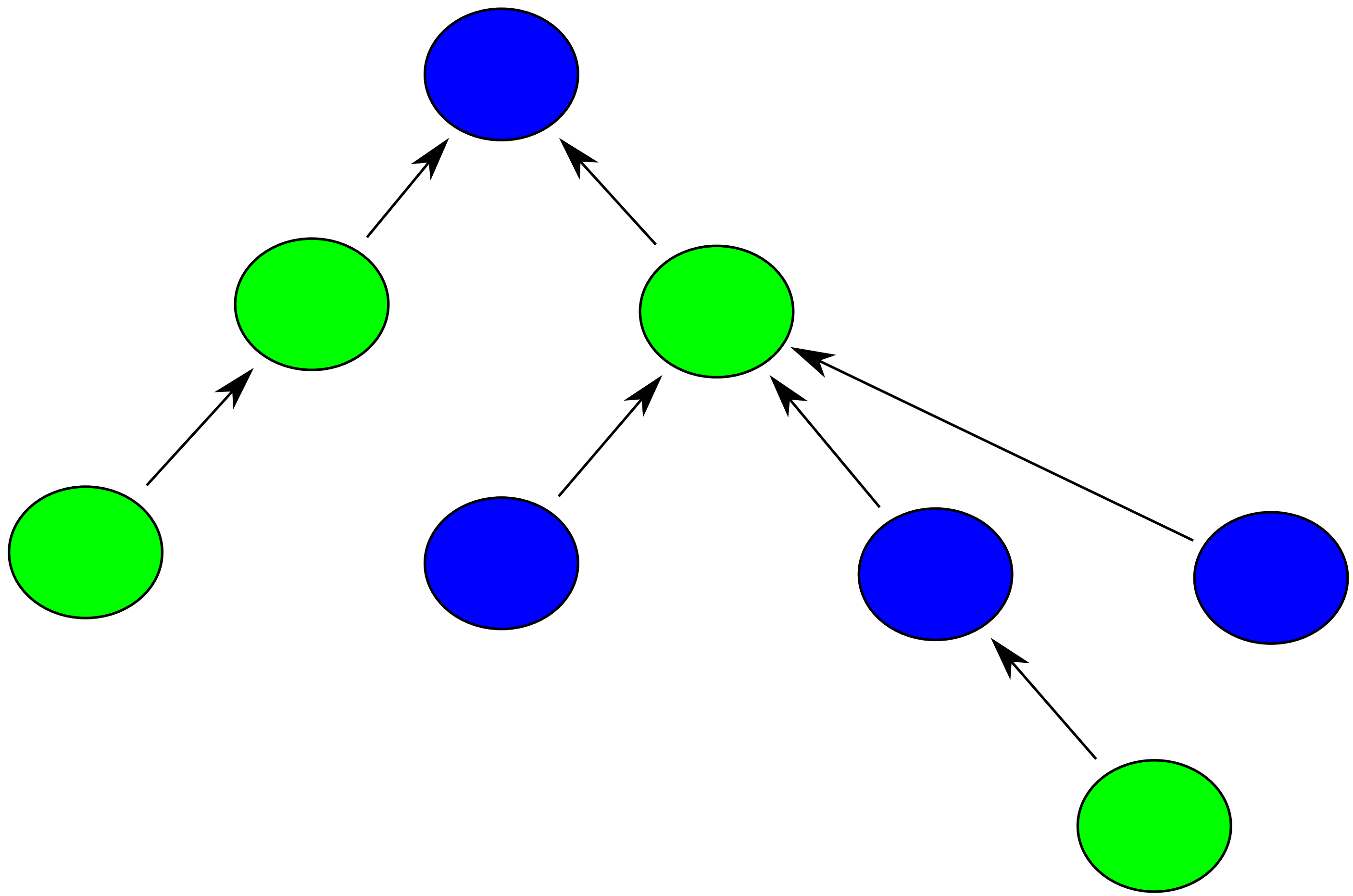
1-backtracking game G^1 of G

Eloise's turn



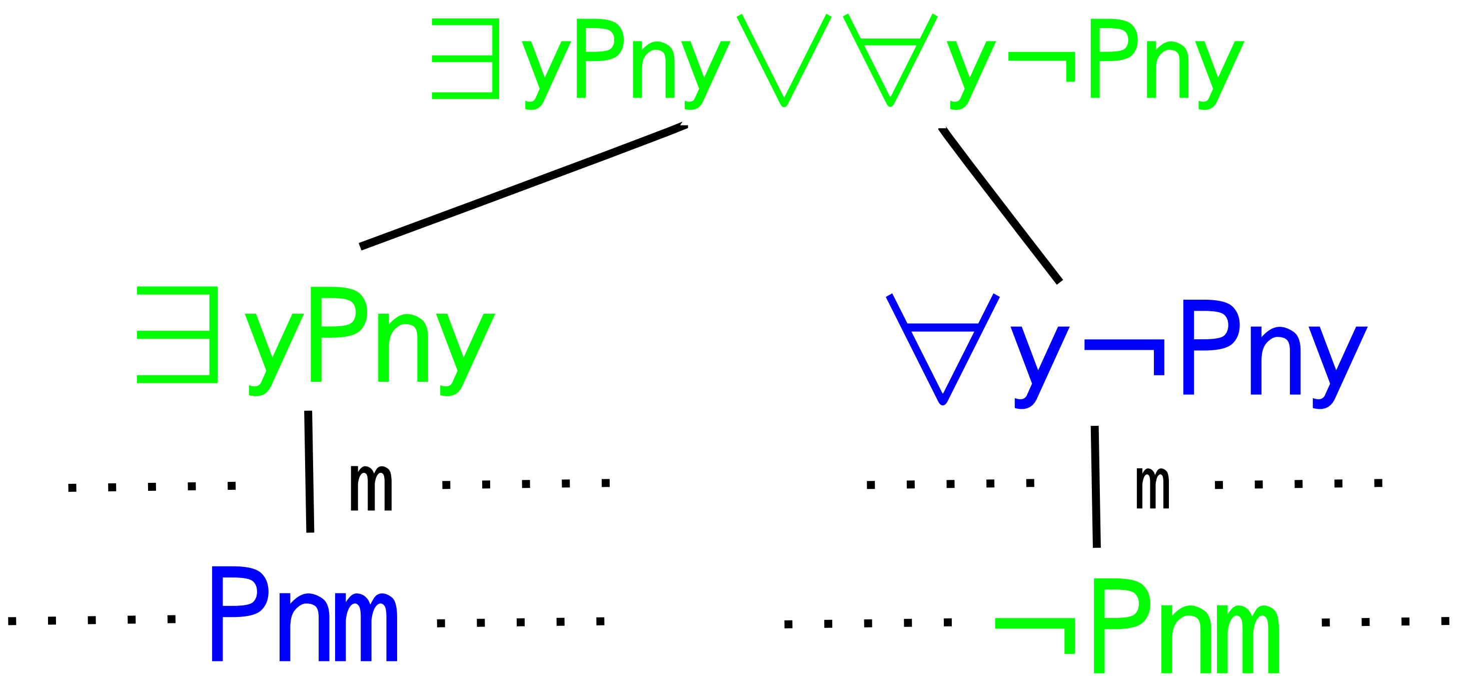
1-backtracking game G^1 of G

Eloise can attach a move any position in the rightmost-path



Tarski game

Tarski game is a important family of games in Logic.



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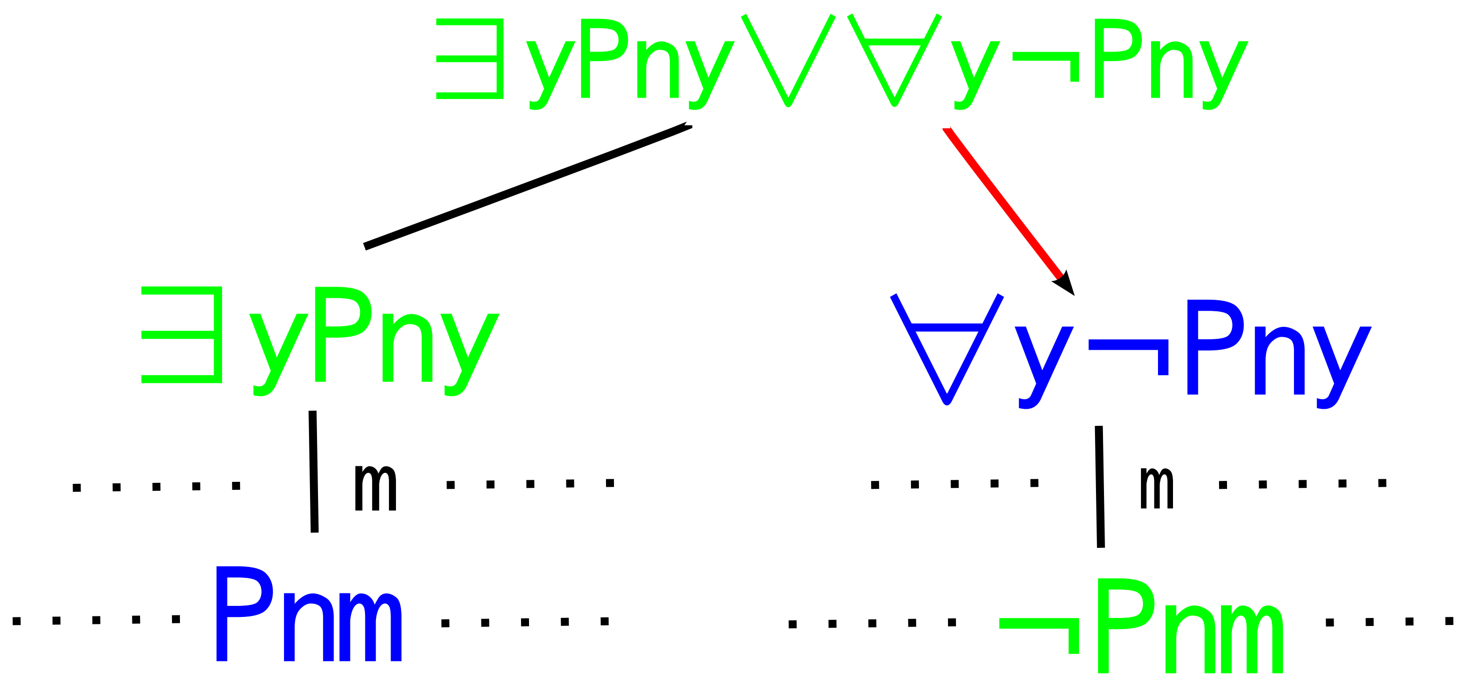

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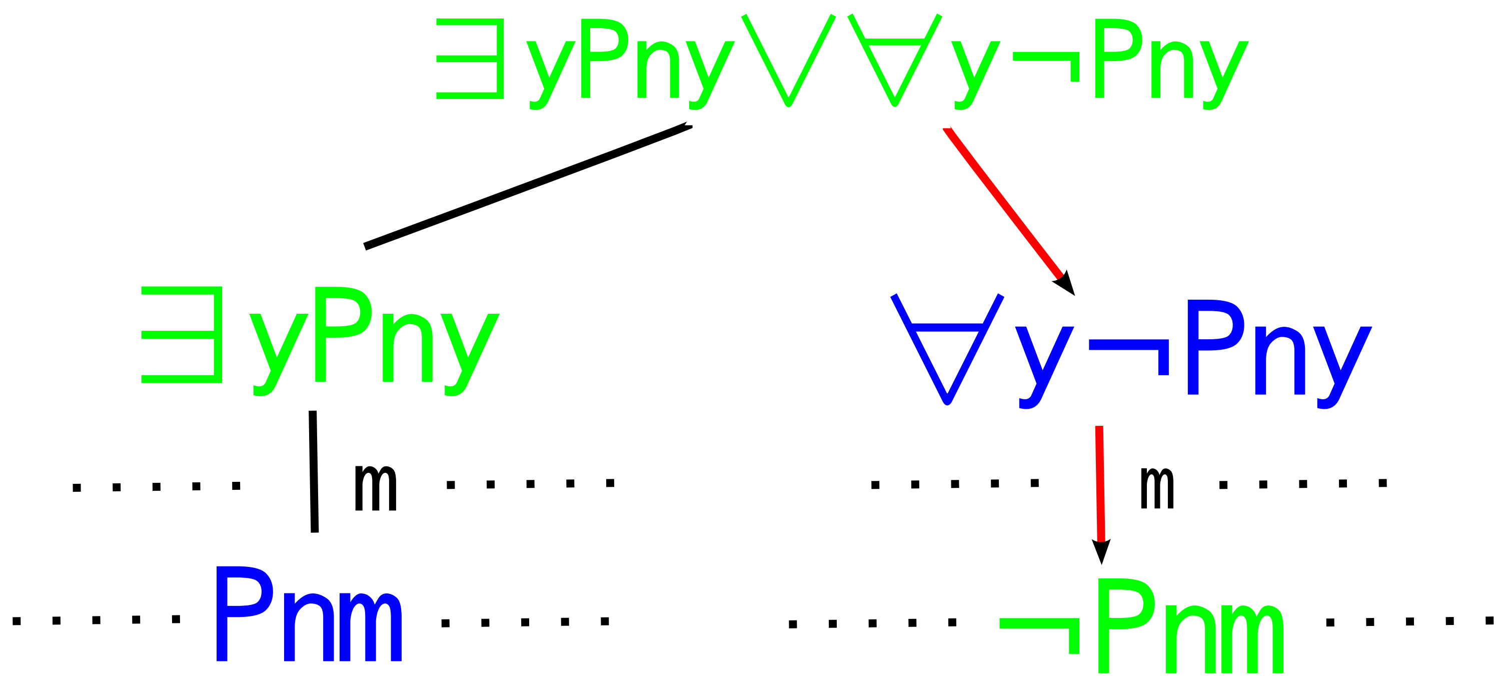


Eloise looses here.

Known Fact

FACT: Eloise has a recursive winning strategy for G^1.

- EM_1 be a schema $\exists y Pxy \vee \forall y \neg Pxy$ (P : decidable).
- G be Tarski game over EM_1.

C.F. : Eloise does not have a recursive winning strategy for G in general.

1-backtracking game of EM_1

$$\exists y P n y \lor \forall y \neg P n y$$

1-backtracking game of EM_1

$$\exists y P n y \vee \forall y \neg P n y$$

$$\forall y \neg P n y$$

1-backtracking game of EM_1

$\exists y Pny \vee \forall y \neg Pny$

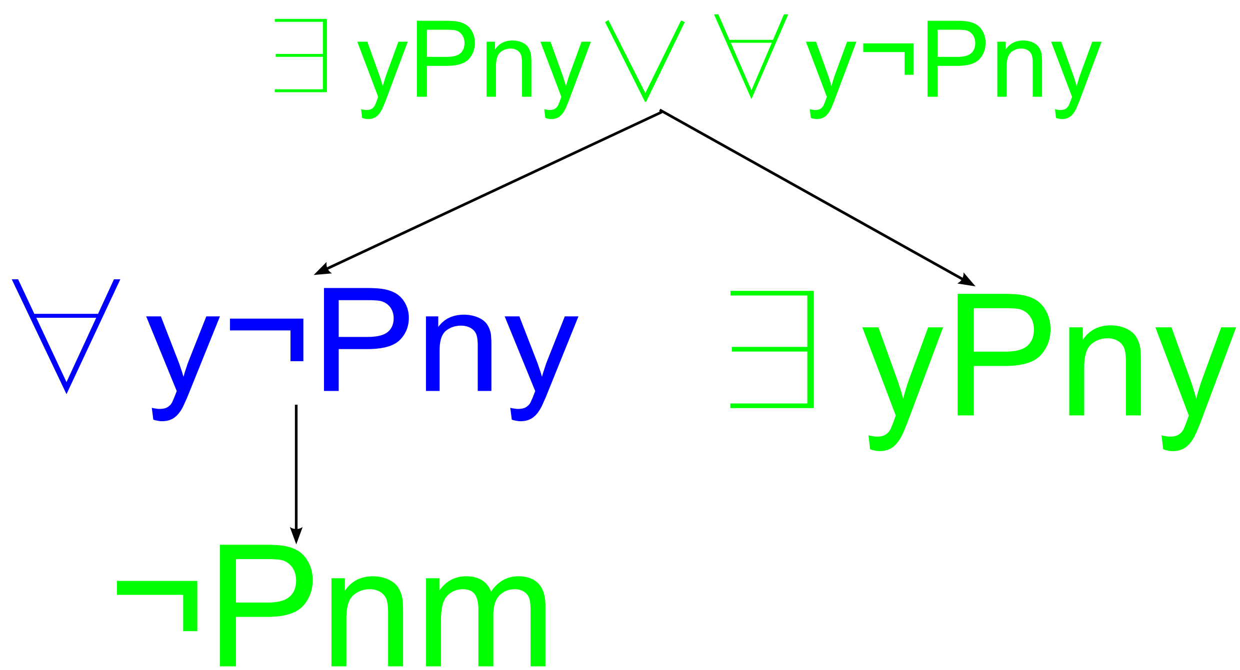
↓

$\forall y \neg Pny$

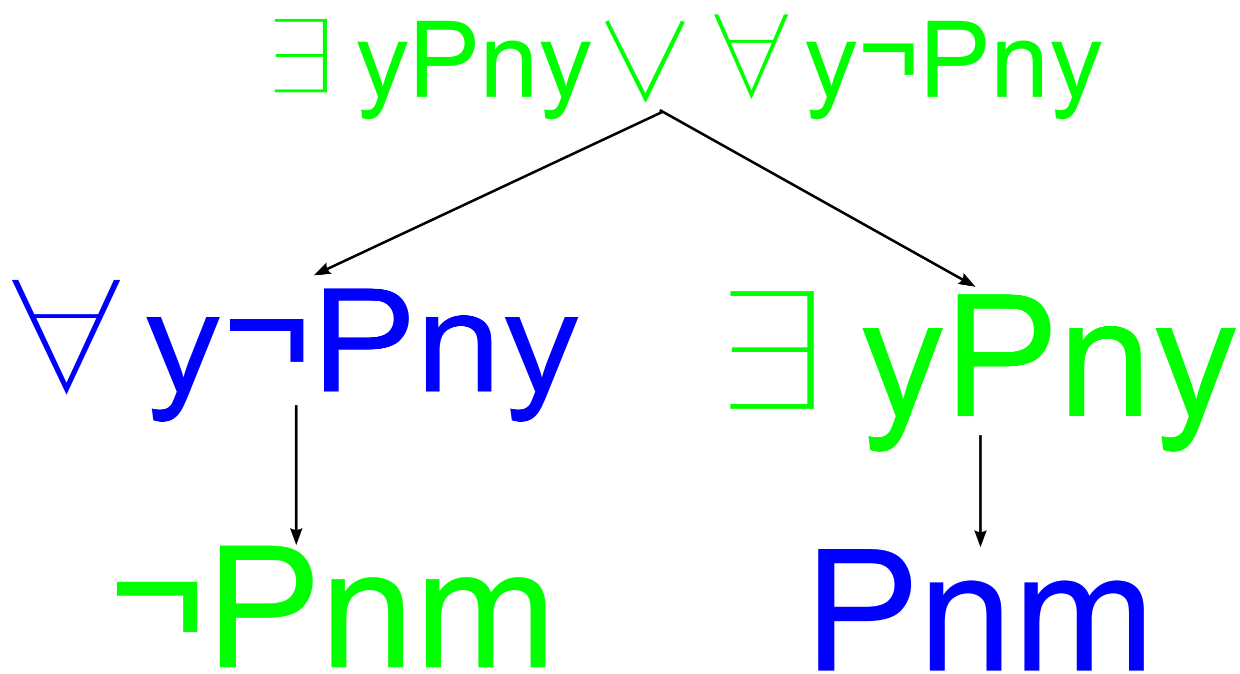
↓

$\neg Pnm$

1-backtracking game of EM_1



1-backtracking game of EM_1



Eloise wins here.

More Known Fact

For positive formula A and its Tarski game T_A, followings are equivalent

- Eloise has a recursive winning strategy for T_A^1
- A has recursive proof in ω-logic without Exchange
- A has recursive proof in intuitionistic logic + EM_1
- A is valid in Limit Computable Mathematics(LCM)

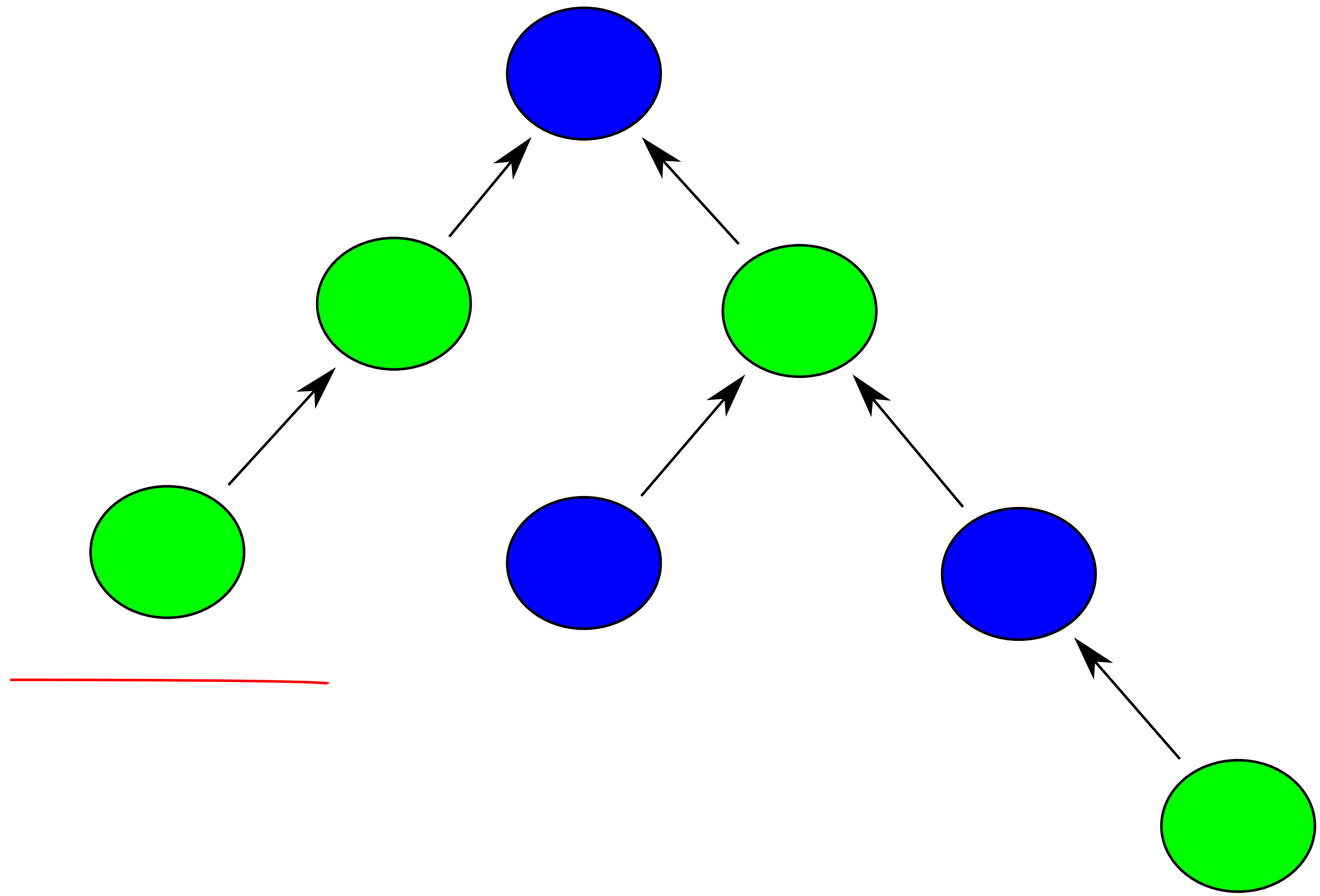
It is known that LCM formalise large parts of Algebra.

Part 2

2-backtracking game

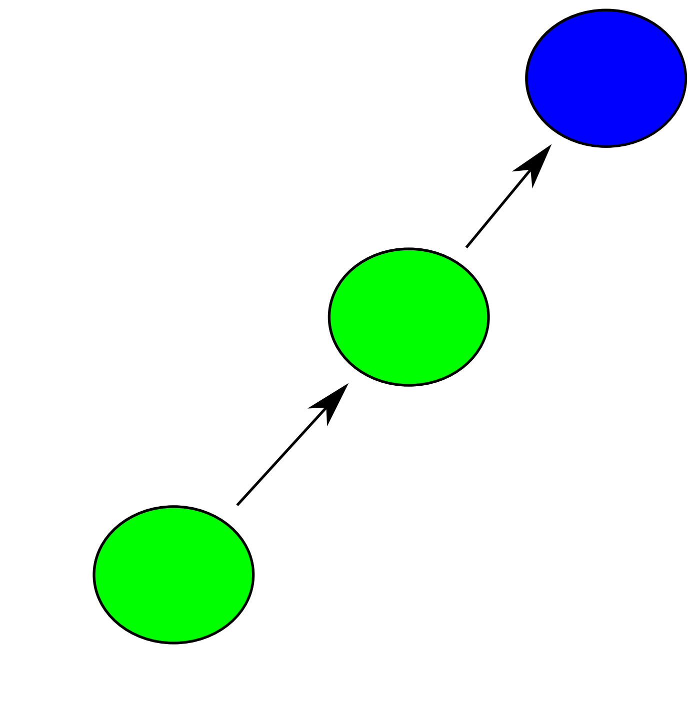
2-backtracking game G^2 of G

Eloise now can choose any E-move in the position tree,



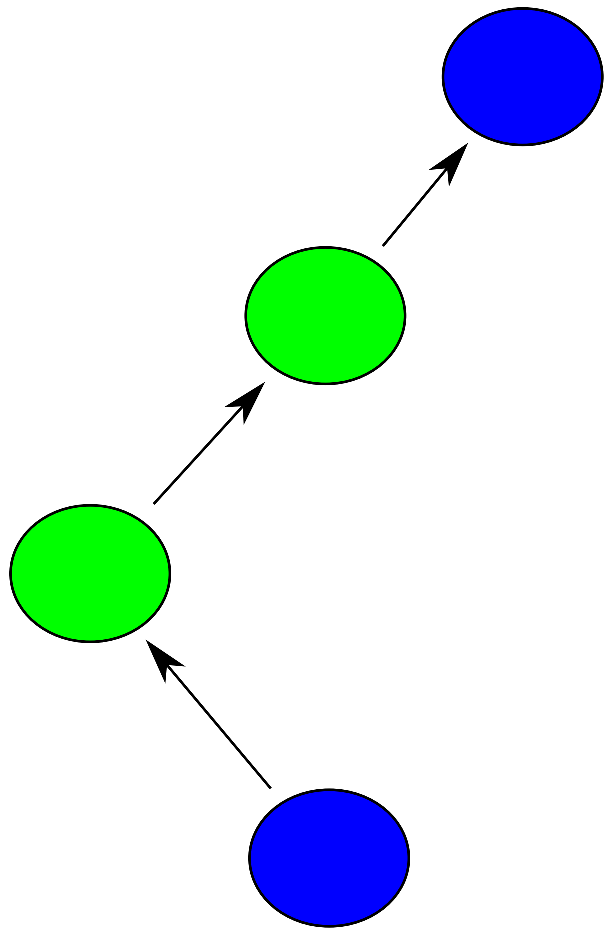
2-backtracking game G^2 of G

remove the moves "after" that,



2-backtracking game G^2 of G

and attach the move to it.



Fact

FACT: Eloise has a recursive winning strategy for G^2.

- EM_2 be a schema

$$\exists y \forall z Q x y z \vee \forall y \exists z \neg Q x y z$$

  (Q : decidable).
- G be Tarski game over EM_2.

C.F. : This does not hold for G^1 in general.

2-backtracking game of EM_2

$$\exists y \forall z Qnyz \vee \forall y \exists z \neg Qnyz$$

2-backtracking game of EM_2

$\exists y \forall z Qnyz \vee \forall y \exists z \neg Qnyz$

$\forall y \exists z \neg Qnyz$

2-backtracking game of EM_2

$$\exists y \forall z Qnyz \vee \forall y \exists z \neg Qnyz$$

$$\forall y \exists z \neg Qnyz$$

$$\exists z \neg Qnmz$$

2-backtracking game of EM_2

$$\exists y \forall z Qnyz \vee \forall y \exists z \neg Qnyz$$

$$\forall y \exists z \neg Qnyz \qquad \exists y \forall z Qnyz$$

$$\exists z \neg Qnmz$$

2-backtracking game of EM_2

$$\exists y \forall z Qnyz \vee \forall y \exists z \neg Qnyz$$

$$\forall y \exists z \neg Qnyz \qquad \exists y \forall z Qnyz$$

$$\exists z \neg Qnmz \qquad \forall z Qnmz$$

2-backtracking game of EM_2

$$\exists y \forall z Qnyz \vee \forall y \exists z \neg Qnyz$$

$$\forall y \exists z \neg Qnyz \qquad \exists y \forall z Qnyz$$

$$\exists z \neg Qnmz \qquad \forall z Qnmz$$

$$Qnmt$$

2-backtracking game of EM_2

$$\exists y \forall z Qnyz \vee \forall y \exists z \neg Qnyz$$

$$\forall y \exists z \neg Qnyz$$

$$\exists z \neg Qnmz$$

$$Qnmt$$

Eloise wins here.

Fact on EM_2

EM_2 implies

- Weak König Lemma
- Bolzano-Weierstrass theorem
- Hence, large parts of analysis

Question

Can we develop a formal system for 2-backtracking game?

Part 3

CL(2) - current attempt

CL(2)

CL(2) : proofs in classical logic satisfying *subgoal-restriction*.

We conjecture CL(2) corresponds 2-backtracking game.

Difficulty

- We need correspondence between trees and sequents
- Hence, we introduce a tree structure on sequents.

Language

Formulas A are defined recursively as follows.

- $\mathbf{t}$, $\mathbf{f}$ are formulas.
- $\bigwedge_{i\in I} A_i$, $\bigvee_{i\in I} A_i$
  - I : recursively enumerable set
  - $(A_i)_{i\in I}$: recursive family of formulas

CL - Classical Logic

sequent : *list* (not multi-set) of formulas.

- true-rule
- conjunctive-rule
- disjunctive-rule
- Weakening-rule

CL : true-rule

$$\frac{}{\vdash \Gamma, \mathbf{t}}\ \text{true}$$

A sequent containing true formulas is immediately justified.

CL:Conjunctive-rule

$$\frac{\ldots \quad \vdash \Gamma, A_i \quad \ldots \text{(for all } i \in I\text{)}}{\vdash \Gamma, \bigwedge_{i \in I} A_i} \wedge$$

CL:Disjunctive-rule

$$\frac{\vdash \Gamma, \bigvee_{i \in I} A_i, \Pi, A_j}{\vdash \Gamma, \bigvee_{i \in I} A_i, \Pi,} \vee_{(j,n)}$$

where $j \in I$ and the last formula of Π is disjunctive or Π is empty. Furthermore, n is the number of the formula of Γ.

CL:Weakening-rule

$$\frac{\vdash \Gamma}{\vdash \Gamma, \Pi}\ W$$

where Π is not empty and the last formulas of Π are disjunctive.

Subgoal-structure

- Tree structure attached each sequent in the proof
- Subgoal-structure is defined only for proofs with a single formula as a conclusion.

Subgoal-structure

For the conclusion, subgoal structure is defined as a tree consisting a single node whose label is the conclusion.

Subgoal-structure

For conjunctive rules

$$\frac{\ldots \quad \vdash \Gamma, A_i \quad \ldots \text{(for all } i \in I)}{\vdash \Gamma, \bigwedge_{i\in I} A_i} \wedge$$

Each Γ, A_i “inherits” the same subgoal structure to $\bigwedge_{i\in I} A_i$, in the sense that subgoal structure of Γ, A_i is obtained by replacing the occurrence of $\bigwedge_{i\in I} A_i$ in the subgoal structure of $\Gamma, \bigwedge_{i\in I} A_i$ to A_i.

Subgoal-structure

For disjunctive rules

$$\frac{\vdash \Gamma, \bigvee_{i\in I} A_i, \Pi, A_j}{\vdash \Gamma, \bigvee_{i\in I} A_i, \Pi} \vee_{(j,n)}$$

we add new node for A_i to subgoal structure of $\vdash \Gamma, \bigvee_{i\in I} A_i, \Pi$ under $\bigvee_{i\in I} A_i$, and obtain subgoal structure for $\Gamma, \bigvee_{i\in I} A_i, \Pi, A_j$.

Subgoal-restriction

All disjunctive rules

$$\frac{\vdash \Gamma, \bigvee_{i \in I} A_i, \Pi, A_j}{\vdash \Gamma, \bigvee_{i \in I} A_i, \Pi} \; \vee_{(j,n)}$$

satisfies that in the subgoal structure of $\Gamma, \bigvee_{i \in I} A_i, \Pi$, all formulas in Π are located under $\bigvee_{i \in I} A_i$.

Conjecture

If A is provable in a proof which satisfies subgoal-restriction

iff

Eloise has a winning strategy of T_A^2

Conclusion

- Review the results of 1-backtracking games
- Introduce CL(2)
- Make a conjecture stating CL(2) characterizes 2-backtracking.