Chiral sine-Gordon model

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We investigate the chiral sine-Gordon model using the renormalization group method. The chiral sine-Gordon model is a model for G-valued fields and describes a new class of phase transitions, where G is a compact Lie group. We show that the model is renormalizable by means of a perturbation expansion and we derive beta functions of the renormalization group theory. The coefficients of beta functions are represented by the Casimir invariants. The model contains both asymptotically free and ultraviolet strong coupling regions. The beta functions have a zero which is a bifurcation point that divides the parameter space into two regions; they are the weak coupling region and the strong coupling region. A large-N model is also considered. This model is reduced to the conventional sine-Gordon model that describes the Kosterlitz-Thouless transition near the fixed point. In the strong-coupling limit, the model is reduced to a $U(N)$ matrix model.

Introduction The chiral model was generalized to the Wess-Zumino-Witten (WZW) model by including the Wess-Zumino term.[1–5]. The WZW model gives a model of conformal field theory whose current algebra is realized by a Kac-Moody algebra. The massive chiral model, which is the chiral Lagrangian[6, 7] or the non-linear sigma model with the mass term, is also interesting as a two-dimensional field theory. The massive chiral model can be regarded as a generalization of the sine-Gordon model to the model for G-valued fields[8], where we add the term of type $\mathrm{Tr}(g + g^{-1})$ for $g \in G$ where G is a general gauge group (Lie group).

The sine-Gordon model has universality and appears in various fields of physics[9–12]. The two-dimensional (2D) sine-Gordon model describes the Kosterlitz-Thouless transition of the 2D classical XY model[13, 14]. The 2D sine-Gordon model is mapped to the Coulomb gas model with logarithmic interaction[15]. The Kondo problem also belongs to the same universality class where the scaling equations are given by those for the 2D sine-Gordon model[16]. The one-dimensional Hubbard model is also mapped onto the 2D sine-Gordon model by using a bosonization method[17, 18]. The sine-Gordon model plays an important role in superconductors, especially multi-band superconductors[19–24] including layered high-temperature superconductors. Generalized sine-Gordon models have also been investigated[25–27].

In this paper we investigate the G-valued nonabelian sine-Gordon model. We consider compact Lie groups such as $G = SU(N)$. The nonabelian sine-Gordon model is renormalizable as for the $U(1)$ sine-Gordon model. Although the chiral model shows an asymptotic freedom in two dimensions, it is lost by the mass term in general. The beta functions, however, have zero at a critical point and this point is a bifurcation point that divides the parameter space into two regions; one is the weak coupling region and the other is the strong coupling region. The asymptotic freedom is realized in the weak coupling region. In the strong coupling limit, the $SU(N)$

sine-Gordon model is reduced to a unitary matrix model. It has been shown by Gross and Witten that in the large N limit there is a third-order transition at some critical coupling constant[28]. Brezin and Gross generalized the coupling constant to be a matrix and found that there is also a phase transition[29–31].

This paper is organized as follows. In Section II, we show the action of the G-valued chiral sine-Gordon model. In section III, we present the renormalization group method in the minimal subtraction scheme using the dimensional regularization method[32–34]. We discuss the renormalization flow in Section IV. In the subsequent section we discuss a relation to the Kosterlitz-Thouless transition for large N. We also examine a relationship with a unitary Matrix model in the strong coupling limit. We give a discussion on the relevance of our model to some problems and give a summary in the last Section.

Chiral sine-Gordon model The sine-Gordon model is given by[11, 15]

$$\mathcal{L} = \frac{1}{2t}\left(\partial_\mu \varphi\right)^2 + \frac{\alpha}{t}\cos\varphi, \tag{1}$$

where φ is a real scalar field. This Lagrangian is written as, by defining $g = e^{i\varphi} \in U(1)$,

$$\mathcal{L} = \frac{1}{2t}\partial_\mu g \partial^\mu g^{-1} + \frac{\alpha}{2t}(g + g^{-1}). \tag{2}$$

This is generalized to a general Lie group G:

$$\mathcal{L} = \frac{1}{2t}\mathrm{Tr}\partial_\mu g \partial^\mu g^{-1} + \frac{\alpha}{2t}\mathrm{Tr}(g + g^{-1}), \tag{3}$$

for $g \in G$. We adopt that $t > 0$ and $\alpha > 0$. This model can be regarded as a chiral model with the potential term of sine-Gordon type. In this paper we consider the chiral sine-Gordon model and derive the renormalization group equation. The renormalization group equation for the sine-Gordon model was derived by using the Wilson method[35] or the perturbation method[15]. In this paper we use the perturbation in terms of t and α.

Renormalization Group Theory *Chiral Lagrangian* An element g of the Lie Group G is represented in the form:

$$g = g_0 \exp(i\lambda T_a \pi_a), \qquad (4)$$

where λ is a real number $\lambda \in \mathbf{R}$ and g_0 is a some element in G. Repeated indices imply the summation is to be done. $\{T_a\}$ $(a = 1, 2, \cdots, N_T)$ is a basis set of the Lie algebra $\mathbf{g}$ where $\mathbf{g}$ is the Lie algebra of G. $N_T = N^2 - 1$ for $G = SU(N)$. π_a $(a = 1, 2, \cdots, N_T)$ are scalar fields. λ is introduced as an expansion parameter and the results do not depend on λ. Thus we can put $\lambda = 1$. $\{T_a\}$ satisfy

$$[T_a, T_b] = i f_{abc} T_c, \qquad (5)$$

where $\{f_{abc}\}$ are structure constants and are totally antisymmetric. $\{T_a\}$ are normalized as

$$\mathrm{Tr} T_a T_b = c \delta_{ab}, \qquad (6)$$

for a real constant c. The normalization constant can take on any real number. For example, for $G = SU(2)$ we can take $T_a = \sigma_a/2$ (Pauli matrices) with $c = 1/2$.

The renormalization of the coupling constant t comes from the kinetic term and the mass term. The former is missing for the conventional $(U(1))$ sine-Gordon model. Let us first consider the renormalization of the kinetic term, namely, the chiral Lagrangian given as

$$\mathcal{L}_{chiral} = \frac{1}{2t_0} \mathrm{Tr} \partial_\mu g \partial^\mu g^{-1}, \qquad (7)$$

where t_0 is the bare coupling constant. We define the renormalized coupling constant t in the following way:

$$t_0 = t \mu^{2-d} Z_t^{-1}, \qquad (8)$$

where Z_t is the renormalization constant. We can take g_0 to be an arbitrary solution of the classical field equation. As a special case we can set $g_0 = 1$ (unit matrix). We introduce the renormalization constant of the field π_a:

$$\pi_{a,B} = \sqrt{Z_\pi} \pi_a. \qquad (9)$$

We expand g by means of π_a as

$$g = g_0 \left[1 + i\lambda T_a \pi_a - \frac{1}{2}\lambda^2 (T_a \pi_a)^2 + \cdots \right]. \qquad (10)$$

The renormalization of the chiral model was investigated by Witten[3]. The correction to the term $(1/2t_0)\mathrm{Tr}\partial_\mu g_0 \partial^\mu g_0^{-1}$ is

$$\Delta\mathcal{L} = -\mathrm{Tr}\partial_\mu g_0 \partial^\mu g_0^{-1} \frac{C_2(G)}{8dc} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m_0^2}, \qquad (11)$$

where $C_2(G)$ is the Casimir invariant in the adjoint representation defined by $\sum_{ab} f_{abc} f_{abd} = C_2(G)\delta_{cd}$:

$$C_2(G) = 2Nc \quad \text{for} \quad G = SU(N). \qquad (12)$$

$C_2(G)$ is proportional to $(N - 2)c$ for $G = O(N)$. m_0 is the mass to avoid the infrared singularity. The results are

$$\beta_t(\mu) \equiv \mu \frac{\partial t}{\partial \mu} = (d - 2)t - \frac{C_2(G)}{8c} t^2 \frac{\Omega_d}{(2\pi)^d}, \qquad (13)$$

for

$$Z_t = 1 + \frac{C_2(G)}{8c} t \frac{1}{\epsilon} \frac{\Omega_d}{(2\pi)^d}. \qquad (14)$$

Here $d = 2 - \epsilon$ and $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the solid angle in d-dimensional space.

When g_0 is a constant matrix, we should expand g up to the third order of π_a to obtain the correction to the kinetic term of π_a[15]. Then we obtain the beta function β_t in a similar form[15]. This means that the two-point function of π_a contains the term p^2/ϵ.

Renormalization of the potential term Let us turn to the potential term $\mathrm{Tr}(g + g^{-1})$. The renormalization is performed by expanding g:

$$\mathrm{Tr} g = \mathrm{Tr} g_0 \left[1 + i\lambda T_a \pi_a - \frac{1}{2}\lambda^2 T_a T_b \pi_a \pi_b + \cdots \right]. \qquad (15)$$

The renormalization of α is evaluated as

$$\begin{aligned}
\alpha_0 \mathrm{Tr} g &\simeq \alpha_0 \mathrm{Tr} g_0 \left[1 - \frac{1}{2}\lambda^2 T_a^2 \langle \pi_a \pi_a \rangle + \cdots \right] \\
&= \alpha_0 \mathrm{Tr} g_0 \left[1 - \frac{1}{2c} t T_a^2 \frac{1}{\epsilon} \frac{\Omega_d}{(2\pi)^d} + \cdots \right],
\end{aligned} \qquad (16)$$

up to the order of t. The bare coupling constant α_0 is related to the renormalized α as follows.

$$\alpha_0 = \alpha \mu^2 Z_\alpha^{-1}. \qquad (17)$$

Then the renormalization constant Z_α is determined as

$$Z_\alpha = 1 - \frac{t}{2c} T_a^2 \frac{1}{\epsilon} \frac{\Omega_d}{(2\pi)^d} = 1 - \frac{t}{2c} C(N) \frac{1}{\epsilon} \frac{\Omega_d}{(2\pi)^d}, \qquad (18)$$

where $C(N) \cdot I \equiv \sum_a T_a^2$ is the Casimir invariant in the fundamental representation (I is the unit matrix) which is given by

$$C(N) = c \frac{N^2 - 1}{N} \quad \text{for} \quad G = SU(N), \qquad (19)$$

$$= c \frac{N - 1}{2} \quad \text{for} \quad G = O(N), \qquad (20)$$

The beta function for α is

$$\mu \frac{\partial \alpha}{\partial \mu} = -2\alpha + \frac{1}{2c} t \alpha C(N) \frac{\Omega_d}{(2\pi)^d}. \qquad (21)$$

The Casimir invariant in the fundamental representation appears in the equation of α.

Renormalization of the two-point function There is a contribution to the renormalization of the coupling constant t from the mass term. We consider this up to the
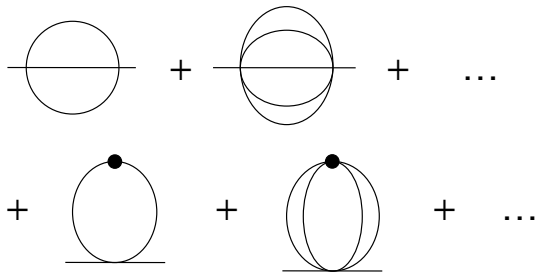
FIG. 1: Contributions to the two-point function.

second order of α. We set $g_0 = 1$ for simplicity. We introduce the renormalization constant of the field π_a:

$$\pi_{a,B} = \sqrt{Z_\pi}\pi_a. \tag{22}$$

Using the expansion of g by means of π_a, we consider the terms that contribute to the two-point function as in the method for the $U(1)$ sine-Gordon model[36]. The diagrams that contribute to the two-point function are shown in Fig.1.

First, we define the Green's function:

$$
\begin{aligned}
G_0(\mathbf{x}) &= \lambda^2 \langle \pi_a(\mathbf{x})\pi_a(0) \rangle \\
&= \frac{1}{c} t\mu^{2-d} Z_t^{-1} Z_\pi^{-1} \int \frac{d^d p}{(2\pi)^d} \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{p^2 + m_0^2} \\
&= \frac{1}{c} t\mu^{2-d} Z_t^{-1} Z_\pi^{-1} \frac{\Omega_d}{(2\pi)^d} K_0(m_0 x),
\end{aligned} \tag{23}
$$

where $x = |\mathbf{x}|$ and K_0 is the 0-th modified Bessel function. m_0 is introduced to avoid the infrared divergence. The first term in Fig.1 shows the contraction between $\mathrm{Tr}(\sum_a T_a \pi_a i(x))^4$ and $\mathrm{Tr}(\sum_b T_b \pi_b(y))^4$. This term gives $G_0(\mathbf{x})^3$ and its coefficient is given by $\sum_{bcd} \mathrm{Tr}T_a T_b T_c T_d \times \mathrm{Tr}(T_a T_b T_c T_d)$ where $\mathrm{Tr}(T_a T_b T_c T_d)$ includes the trace of permutations of T_a, T_b, T_c and T_d. We estimate this by using the relation for $SU(N)$ given by

$$T_a T_b = \frac{1}{2N}\delta_{ab}\cdot I + \frac{1}{2}\sum_{c=1}^{N^2-1}(if_{abc}+d_{abc})T_c, \tag{24}$$

for $c = 1/2$. $\{d_{abc}\}$ are structure constants and are totally symmetric with respect to indices a, b and c. Because we include the permutations of $T_a T_b T_c T_d$, we drop f_{abc} term in this formula. We find that the coefficient of the first term in Fig.1 is of order N^2 for large N by using the relation

$$\sum_{ab} d_{abc}d_{abd} = (N^2-4)/N \cdot \delta_{cd}. \tag{25}$$

Since $\sinh I - I = I^3/3! + \cdots$, the diagrams in Fig.1 are summed up to give[36]

$$\Sigma = \int d^d x \Big[e^{i\mathbf{p}\cdot\mathbf{x}}(\sinh I - I) - (\cosh I - 1) \Big], \tag{26}$$

where $I = C(N)G_0(x)$. We use $\sinh I - I \approx (1/2)e^I$ and $\cosh I \approx (1/2)e^I$ for large I. We included $C(N)$ in I. We define $A_0 \equiv \sum_{bcd} \mathrm{Tr}T_a T_b T_c T_d \times \mathrm{Tr}(T_a T_b T_c T_d)/C(N)^3$ which is of order $1/N$ for large N. Then, the correction to the bare two-point function for π fields is given as

$$
\begin{aligned}
\Gamma_B^{(2)c}(p) = &-\frac{1}{2}\left(\frac{\alpha}{t}\mu^d Z_\alpha^{-1} Z_t\right)^2 A_0 \frac{t\mu^{2-d}}{Z_t Z_\pi} \int d^d x \left(e^{i\mathbf{p}\cdot\mathbf{x}}-1\right) \\
&\times e^{C(N)G_0(x)}.
\end{aligned} \tag{27}
$$

We use $e^{i\mathbf{p}\cdot\mathbf{x}} = 1 + i\mathbf{p}\cdot\mathbf{x} - (1/2)(\mathbf{p}\cdot\mathbf{x})^2 + \cdots$ and keep the $\mathbf{p}^2$ term. We use the asymptotic formula $K_0(x) \sim -\gamma - \log(x/2)$ for small x. Because t has a fixed point t_c from a zero of eq.(21), we denote by v the deviation of t from the fixed point:

$$\frac{1}{8\pi c}C(N)t = 1 + v, \tag{28}$$

where we set $d = 2$ for the volume element $\Omega_d/(2\pi)^d$. The bare two-point function reads

$$
\begin{aligned}
\Gamma_B^{(2)c}(p) = &\frac{1}{8}A_0 \frac{\alpha^2}{t}\mu^{d+2}p^2 (c_0 m_0^2)^{-2-2v} \Omega_d \frac{Z_t}{Z_\pi} \\
&\times \int_0^\infty dx\, x^{d+1} \frac{1}{(x^2+a^2)^{2+2v}} \\
= &\frac{1}{8}A_0 p^2 \frac{\alpha^2}{t}\mu^{d+2}(c_0 m_0^2)^{-2}\frac{Z_t}{Z_\pi}\Omega_d \frac{1}{\epsilon} \\
\simeq &\, p^2 \frac{1}{32c}A_0 C(N)\alpha^2 \mu^{d+2}(c_0 m_0^2)^{-2}\frac{Z_t}{Z_\pi}\frac{1}{\epsilon} + O(v),
\end{aligned} \tag{29}
$$

where c_0 is a constant and $a = 1/\mu$ is a small cutoff to keep away from a infrared singularity. The divergence of α was absorbed by Z_α. The renormalized two-point function is given by

$$
\begin{aligned}
\Gamma_R^{(2)}(p) &= Z_\pi \Gamma_B^{(2)}(p) \\
&= \frac{cZ_t Z_\pi}{t\mu^{2-d}}\Big[p^2 + p^2 \frac{1}{32c}A_0 C(N)\alpha^2 \mu^{d+2} \\
&\quad \times (c_0 m_0^2)^{-2}\frac{1}{\epsilon}\Big].
\end{aligned} \tag{30}
$$

The divergence is removed by Z_t. For $g_0 = 1$ (constant matrix), $\Gamma_R^{(2)}(p)$ contains the term p^2/ϵ coming from the kinetic part of the Lagrangian. The α^2-term of Z_t is determined as

$$Z_t = 1 - \frac{1}{32c}A_0 C(N)\alpha^2 \mu^{d+2}(c_0 m_0^2)^{-2}\frac{1}{\epsilon}. \tag{31}$$

From the equation $\mu\partial t_0/\partial\mu = 0$ for $t_0 = Z_t^{-1}t\mu^{2-d}$, we obtain the contribution of order α^2 as

$$
\begin{aligned}
\mu\frac{\partial t}{\partial\mu} = &(d-2)t - \frac{1}{32c}A_0 C(N)(c_0 m_0^2)^{-2}t\frac{1}{\epsilon} \\
&\times \left(2\alpha\mu^{d+2}\mu\frac{\partial\alpha}{\partial\mu} + (d+2)\alpha^2\mu^{d+2}\right).
\end{aligned} \tag{32}
$$

With use of eq.(21) this results in

$$\mu\frac{\partial t}{\partial \mu} = (d-2)t + \frac{1}{32c}A_0 C(N)\mu^{d+2}(c_0 m_0^2)^{-2}t\alpha^2\frac{\Omega_d}{(2\pi)^d}$$
$$+ O(\alpha^2 t^2). \tag{33}$$

We perform the finite renormalization for α in the manner[36] $\alpha \to \alpha c_0 m_0^2 a^2 = \alpha c_0 m_0^2 \mu^{-2}$ because the finite part of $G_0(x \to 0)$ is $G_0(x \to 0) = -(1/2\pi)\ln(m_0/\mu)$. We add the result for the chiral Lagrangian to obtain scaling equations:

$$\mu\frac{\partial t}{\partial \mu} = (d-2)t - \frac{C_2(G)}{8c}t^2 + \frac{1}{32c}A_0 C(N)t\alpha^2, \tag{34}$$

$$\mu\frac{\partial \alpha}{\partial \mu} = -\alpha\left(2 - C(N)\frac{1}{2c}t\right), \tag{35}$$

wherae we include $\Omega_d/(2\pi)^d$ into the definition of t for simplicity.

Renormalization flow and asymptotic freedom

Let us consider the renormalization group flow in two dimensions. Since $A_0 \sim O(1/N)$, we set $A_0 = \tilde{A}_0/N$ and we define $\tilde{\alpha} = \sqrt{\tilde{A}_0/32}\alpha$. Then the equations read

$$\mu\frac{\partial t}{\partial \mu} = (d-2)t - \frac{C_2(G)}{8c}t^2 + \frac{1}{cN}C(N)t\tilde{\alpha}^2, \tag{36}$$

$$\mu\frac{\partial \tilde{\alpha}}{\partial \mu} = -\tilde{\alpha}\left(2 - \frac{C(N)}{2c}t\right). \tag{37}$$

In the following, we write $\tilde{\alpha}$ as α for simplicity.

The beta functions for $SU(N)$ with $N \geq 2$ are

$$\mu\frac{\partial t}{\partial \mu} = (d-2)t - \frac{N}{4}t^2 + \frac{N^2-1}{N^2}t\alpha^2, \tag{38}$$

$$\mu\frac{\partial \alpha}{\partial \mu} = -\alpha\left(2 - \frac{N^2-1}{2N}t\right). \tag{39}$$

This set of equations has several phases depending on the initial values of t and α.

Bifurcation point A set of beta functions has zero at $(t, \alpha) = (t_c, \alpha_c)$ in two dimensions $d = 2$ where

$$t_c = \frac{4c}{C(N)}, \quad \alpha_c = \sqrt{\frac{cNC_2(G)}{2}}\frac{1}{C(N)}. \tag{40}$$

This point, however, divides the parameter space of (t, α) into two regions. One is the strong coupling region and the other is the weak coupling region.

Asymptotic freedom When $\mu\partial t/\partial \mu < 0$ and $\mu\partial\alpha/\partial\mu < 0$, we have an asymptotically free theory. If the parameters (t, α) goes into the region,

$$t < t_c, \quad \frac{C(N)}{N}\alpha^2 < \frac{C_2(G)}{8}t, \tag{41}$$

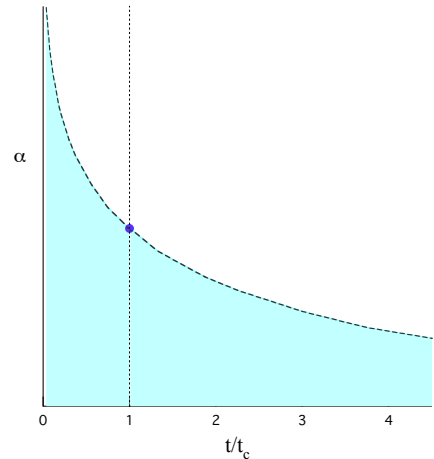t approaches 0 as μ increases. When the flow goes into this region, the model shows an asymptotic freedom. A

FIG. 2: The region (shaded region except for the boundary) where the flow is renormalized into $\mu\partial t/\partial\mu < 0$ and $\mu\partial\alpha/\partial\mu < 0$, so that $t \to 0$ and $\alpha \to 0$ as μ increases. Outside of this region, the flow goes into the strong-coupling region given by $t \to \infty$ and $\alpha \to \infty$ as μ increases. The points on the boundary go to the critical point (t_c, α_c) along the boundary following the renormalization flow.

parameter (t, α) near this region will also go into an asymptotically free region. We show the region of the initial values of (t, α) in Fig.2 where the flows are renormalized into the region of asymptotic freedom with $t \to 0$ and $\alpha \to 0$ as $\mu \to \infty$. Outside of the shaded region, the flow goes to the strong-coupling region, that is, $t \to \infty$ and $\alpha \to \infty$ as μ increases. When the initial parameters (t, α) are on the boundary, they are renormalized into (t_c, α_c).

$SU(2)$ sine-Gordon model We solve a set of scaling equations numerically for the $SU(2)$ case ($N = 2$). In this case we have $(t_c, \alpha_c) = (8/3, 16/3)$. We show the renormalization group flow in Fig.3. Obviously we have the weak coupling and strong coupling regimes. The former is the asymptotically free region where t and α are renormalized to zero. The divide between two regions is shown in Fig.4 by the dashed line. All the renormalization flows approach a line as $\mu \to \infty$, which is shown by the solid line in Fig.4.

In the strong coupling region where t and α are large, the Lagrangian may be approximated by the mass term: $\mathcal{L} \simeq \alpha/(2t)\mathrm{Tr}(g + g^{-1})$. We have mass gap in this region. Instead, in the weak coupling region where both t and α approach zero, the Lagrangian is effectively given by the kinetic term because α is decreased faster than t.

A matrix model and large-N limit In the limit of large t and α/t, the $SU(N)$ sine-Gordon model is reduced to a unitary matrix model. A third-order phase transition has been predicted for this model[28, 29]. Gross and Witten considered the partition function of the form

$$Z = \int dU \exp\left(\beta N \mathrm{Tr}(U + U^\dagger)\right), \tag{42}$$
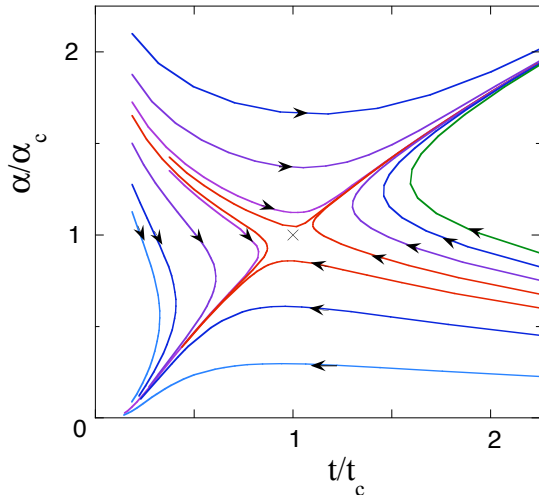
FIG. 3: Renormalization flow as μ increases. The cross indicates the fixed point (t_c, α_c).
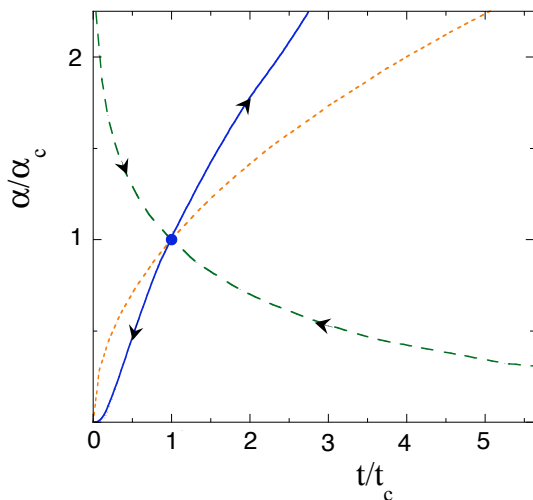

FIG. 4: Renormalization flow approaches the solid line as μ increases. The dotted line indicates $\alpha = \sqrt{2t/3}$ below which we have $\mu \partial t / \partial \mu < 0$. The dashed line represents a ridge that divides the $t - \alpha$ plane into two regions.

where U is, for example, an $N \times N$ special unitary matrix, $UU^\dagger = 1$. There is a phase transition at $\beta = 1/2$ in the large N limit. This is a transition between the weak coupling regime $\beta > 1/2$ and the strong coupling regime $\beta < 1/2$. It has been shown that this is a third-order transition because the third derivative of the free energy, $-\ln Z$, is discontinuous at $\beta = 1/2$.

To consider the relation with the unitary matrix model, we replace α to $N\alpha$ as in eq.(42). When N is large, the $SU(N)$ beta functions are reduced to

$$\mu \frac{\partial t}{\partial \mu} = N^2 t \alpha^2, \quad \mu \frac{\partial \alpha}{\partial \mu} = -\alpha \left(2 - \frac{N}{2} t \right). \quad (43)$$

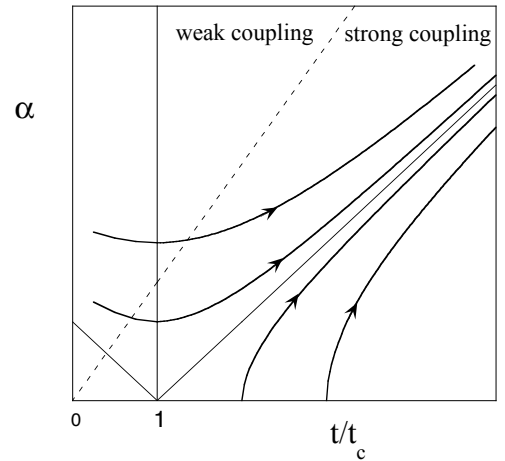The zero of $\mu \partial \alpha / \partial t = 0$ gives the fixed point $t_c \simeq 4/N$.

FIG. 5: The renormalization group flow for large N. The dashed line indicates $\alpha/s = 4$ where $s = t/t_c = 1 + v$.

Near this point, t is parametrized as $t = (4/N)(1 + v)$. Then the equations read

$$\mu \frac{\partial v}{\partial \mu} = \frac{N^2}{2} \alpha^2 + O(v\alpha^2), \quad \mu \frac{\partial \alpha}{\partial \mu} = 2\alpha v, \quad (44)$$

where we neglected the term $v\alpha^2$. When $v\alpha^2$ is small, a set of equations was reduced to that of the Kosterlitz-Thouless transition[13, 15], namely, that of the Abelian $(U(1))$ sine-Gordon model. We define $x = 2v$ and $y = N\alpha$ to obtain[13] $\mu \partial x / \partial \mu = y^2$, $\mu \partial y / \partial \mu = xy$, i.e., the renormalization group flow, which is identical to that of the Kosterlitz-Thouless transition.

We define $s \equiv t/t_c \simeq (N/4)t$. The Lagrangian is written as

$$\mathcal{L} = \frac{N}{8s} \mathrm{Tr} \partial_\mu g \partial^\mu g^{-1} + \frac{N\alpha}{8s} \mathrm{Tr}(g + g^{-1}). \quad (45)$$

In the strong coupling region, $s \gg 1$ and $\alpha \gg 1$, $\mathcal{L}$ is approximated by a matrix model $\mathcal{L}_m = N\alpha/(8s)\mathrm{Tr}(g + g^{-1})$. $\alpha/(8s)$ corresponds to β and $\alpha/s = 4$ is a divide of strong and weak coupling regimes. The renormalization flow in the large N limit is shown in Fig.5, where the flow goes into the strong coupling region as μ increases.

Discussion Our model is a generalization of the Abelian $U(1)$ sine-Gordon model to the model with gauge group G, and is also regarded as a generalization of the chiral model with a mass term. This model is also regarded as a nonabelian generalization of the Josephson model in superconductors. The nonabelian sine-Gordon model is a multi-component field theory and each independent component in g represents a mode in a periodic potential.

When we take $g = e^{-i\varphi\sigma_3/2} e^{i\theta\sigma_2/2} e^{i\varphi\sigma_3/2}$, the Lagrangian is $\mathcal{L} = (1/4t)\left[(\partial_\mu \theta)^2 + 4\sin^2 \theta (\partial_\mu \varphi)^2 \right] + (2\alpha/t)\cos(\theta/2)$. The kinetic term is that for the nonlinear sigma model with

$\mathbf{n} = (\sin\theta\cos(2\varphi), \sin\theta\sin(2\varphi), \cos\theta)$. This model has one massive and one massless mode, if we replace $\sin^2\theta$ by its expectation value using a mean-field like treatment. We can generalize the model to have multiple massive modes and multiple massless Nambu-Goldstone modes. We can add an external field or a higher oder potential term such as $\alpha_2 \text{Tr} g^2$ to the Lagrangian:

$$\mathcal{L} = \frac{1}{4t}\Big[(\partial_\mu\theta)^2 + 4\sin^2\theta(\partial_\mu\varphi)^2\Big] + \frac{2\alpha}{t}\cos\left(\frac{\theta}{2}\right) + \frac{2\alpha_2}{t}\cos\theta. \tag{46}$$

The potential term may have a non-trivial ground state with finite θ depending on the sign of α and α_2. In the state with a finite stationary value of θ, the ground states are degenerate, leading to the breaking of time-reversal symmetry (or CP invariance)[24]. There is a transition as α_2 is varied. We expect that this kind of transition may occur in an unconventional superconductor[37].

For $g = e^{i\phi\sigma_2/2}e^{i\varphi\sigma_3/2} \in SU(2)$, the Lagrangian reads

$$\mathcal{L} = \frac{1}{4t}\left((\partial_\mu\phi)^2 + (\partial_\mu\varphi)^2\right) + \frac{2\alpha}{t}\cos\left(\frac{\phi}{2}\right)\cos\left(\frac{\varphi}{2}\right)$$
$$+ \frac{\alpha_2}{t}(\cos\phi\cos\varphi + \cos\phi + \cos\varphi - 1), \tag{47}$$

where we added an external field $\text{Tr}(g^2 + g^{-2})$. This is a model of coupled scalar fields, where both fields are massive. There would also be a transition when α and α_2 terms are frustrated.

Summary We investigated the scaling property of the chiral sine-Gordon model with G-valued fields for $G = SU(N)$. We derived a set of renormalization group equations for this model, where the coefficients of the beta functions are determined by Casimir invariants of G. There are two regions in the parameter space of t and α: one is the ultraviolet strong-coupling region and the other is the asymptotically free region. The beta functions have zero at $(t, \alpha) = (t_c, \alpha_c)$ where the model has scale invariance. This point divides the parameter space into two regions.

We considered the large-N model. The beta functions in this model are simplified and reduced to those for the Kosterlitz-Thouless transition, that is, the $U(1)$ sine-Gordon model near the critical point. In the strong coupling limit, the $SU(N)$ model is reduced to a matrix model with $U(N)$-value fields. There may be a third-order phase transition at $\alpha/s = 4$ in the large-N limit.

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