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Fluctuation modes in multi-gap superconductors

Takashi Yanagisawa

Abstract We investigate excitation modes in multi-gap superconductors. Multi-gap superconductors have multiple excitation modes. In particular, the Nambu-Goldstone mode, Leggett mode and Higgs modes are important and play an important role in multi-gap superconductors. The multiple-phase invariance in a multi-gap system is partially or totally spontaneously broken in a superconductor. We evaluate the dispersion relation and the mass formulas of these modes by using the functional integral method. The broken multiple-phase invariance leads to a new quantum phase such as the time-reversal symmetry breaking, the emergence of massless modes and fractionally quantized-flux vortices. There is a possibility that half-flux vortices exist in a two-component superconductor in magnetic field.

Keywords multi-gap superconductivity, multi-band BCS model, Nambu-Goldstone mode, Leggett mode, Higgs mode, frustrated-Josephson coupling, time-reversal symmetry breaking, massless mode, fractional-flux vortex, sine-Gordon model

1 Introduction

The study of multi-band superconductors has a long history and is started from works by Moskalenko[1], Suhl et al.[2], Peretti[3] and Kondo[4], as a generalization of the Bardeen-Cooper-Schrieffer (BCS) theory[5] to a multi-gap superconductor. The first observed two-band superconductor is Nb doped $SrTiO_3$[6, 7]. The critical field $H_{c2}(0)$ and the sizable positive curvature of $H_{c2}(T)$ in YNi_2B_2C and $LuNi_2B_2C$ were analyzed within an effective two-band model on the basis of multi-band Eliashberg theory[8]. MgB_2[9] and iron-based superconductors[10] were discovered later. It was pointed out from a theoretical point of view that the sign of gap function depends on the sign of the pair-transfer interaction between two bands, and the signs of two gaps are opposite to each other when the pair-transfer interaction is repulsive.

Electronics and Photonics Research Institute, National Institute of Advanced Industrial Science and Technology, 1-1-1 Umezono, Tsukuba, Ibaraki 305-8568, Japan
E-mail: t-yanagisawa@aist.go.jp

There are many interesting properties in multi-gap superconductors. We show some of them in the following.

(1) Multi-band superconductors have a possibility to exhibit high critical temperature T_c. T_c is always enhanced in the presence of interband interactions for s-wave superconductors. MgB_2[9] and iron-based superconductors[10] are multi-band superconductors with relatively high T_c. We also mention that layered cuprates[11–13] can be regarded as a multi-gap superconductor. The mechanism of superconductivity in high-temperature cuprates has been studied intensively since its discovery[14–21]. The ladder model is also related with a two-gap superconductor[22–24].

(2) Unusual isotope effect has been observed in multi-band superconductors. This depends on the nature of the attractive interaction in the pairing mechanism[25–28] The isotope exponent α of (Ba,K)Fe_2As_2 takes values even in the range of $\alpha < 0$ and $\alpha > 0.5$, depending on the property of glue, especially strength and the range of attractive interactions[27, 28]. It is sometimes difficult to determine the pairing symmetry of a multi-band superconductor. It is still controversial whether the symmetry of the electron pair is $s_\pm$ or s_{++}, or there is a line node in the gap function of iron-based superconductors.

The pairing symmetry of noncentrosymmetric compounds $LaNiC_2$[29, 30] and $LaNiGa_2$[31–33] are also not confirmed yet. It has been suggested that the time-reversal symmetry is broken in these materials by a muon spin relaxation measurement[29]. The non-unitary triplet pairing was proposed theoretically[29], whereas there are experimental data suggesting that the superconductivity in $LaNiC_2$ is BCS-like[34–36]. It is indicated that $LaNiC_2$ is highly correlated material with strong electronic interactions[37].

(3) In N-gap superconductors, the gap functions are written as $\Delta_j = |\Delta_j|e^{i\theta_j}$ for $j = 1, \cdots, N$. The $U(1)^N$ phase invariance at most can be spontaneously broken. The Coulomb repulsive interaction turns the one-phase mode $\Phi = c_1\theta_1 + \cdots + c_N\theta_N$ into a gapped plasma mode. Thus there are at most $N - 1$ modes and they can be low-energy excitation modes in superconductors. These modes are in general massive due to Josephson interactions. There is, however, a possibility that some of these modes become massless Nambu-Goldstone modes when the Josephson couplings are frustrated.

The Josephson couplings between different bands will bring about attractive phenomena; they are (a) time-reversal symmetry breaking (TRSB)[38–50], (b) the existence of massless (gapless) modes[51–57] and low-lying excited states, and (c) the existence of kinks and fractionally-quantized-flux vortices[58–62]. The phase-difference mode between two gaps is sometimes called the Leggett mode[63]. This mode will yield new excitation modes in multi-gap superconductors. The Leggett mode is realized as a Josephson plasma oscillation in layered superconductors.

(4) The amplitude mode of the gap function is represented by a Higgs boson in a superconductor. The effective action is given by the time-dependent Ginzburg-Landau (TDGL) model when the temperature T is near the critical temperature T_c. The TDGL model includes the dissipation effect on the amplitude ond thus the Higgs mode may not be defined clearly. In contrast, at low temperature $T \ll T_c$, the effective action is given by the quadratic form of the Higgs boson and the mass of the Higgs boson is defined definitely.

(5) The existence of fractionally quantized-flux vortices is very significant and interesting. The kink (soliton) solution of phase difference leads to a new mode and the existence of half-quantum flux vortices in two-gap superconductors. A generalization to a three-gap superconductor is not trivial and results in very attractive features, that is, chiral states with time-reversal symmetry breaking and the existence of fractionally quantized vortices[38–40, 42]. Further, in the case with more than four gaps, a new state is predicted with a gapless excitation mode[64].

(6) A new type of superconductors, called the 1.5 type as an intermediate of types I and II, was proposed for two-gap superconductors[65, 66]. The 1.5-type state suggests that an attractive interaction works between vortices. This state may be realized as a result of a multi-band effect, and does not occur in a single-band superconductor.

(7) There is an interesting and profound analogy between particles physics and superconductivity. For example, there is a similarity between the Dirac equation and the gap equation of superconductivity[67, 68]. Nambu first noticed this property and brought the idea of spontaneous symmetry breaking into the particle physics.

The mass of the Higgs particle corresponds to the inverse of the coherence length, and the masses of gauge bosons W and Z correspond to the inverse of the penetration depth. When we use $m_W \sim 80.41\mathrm{GeV}/c^2$, $m_Z \sim 91.19\mathrm{Gev}/c^2$ and $m_H \sim 126\mathrm{GeV}/c^2$, the Ginzburg-Landau parameter κ is roughly

$$\kappa = \frac{\lambda}{\xi} \sim \frac{m_{W,Z}}{m_H} \sim 1.5. \tag{1}$$

This suggests that the universe corresponds to a type-II superconductor.

In this paper we discuss several interesting properties of the Nambu-Goldstone mode, the Leggett mode and the Higgs mode in multi-gap superconductors. We focus on superconductors in the clean limit, and impurity effects are left for future studies. The paper is organized as follows. We show the basic formulation of the BCS theory and discuss an analogy between the theory of superconductivity and the mass generation in the particle physics in Section II. In Section III we give a survey on a history of multi-gap superconductivity. We give a formula for the effective action on the basis of the functional integral method in Section IV. We discuss the Nambu-Goldstone and Leggett modes in Section V. The dispersion relation of the Higgs mode is examined in Section VI. Section VII is devoted to a discussion on time-reversal symmetry breaking. In Section VIII we show that the half-quantized-flux vortex can be regarded as a monopole in a multi-gap superconductor. In Section IX we discuss the emergency of massless Nambu-Goldstone mode when there is a frustration between Josephson couplings. We give a discussion on the sine-Gordon model in Section X. We investigate a chiral symmetry breaking where fluctuations restore time reversal symmetry from the ground state with time-reversal symmetry breaking in the subsequent section XI. In Section XII we show an $SU(N)$ sine-Gordon model.which is a generalization This model is a generalization of the conventional sine-Gordon model to that with multiple variables, and is regarded as a model of G-valued fields for a Lie group G. This model is reduced to a unitary matrix model in some limit. We give a summary in the last section.

2 Gap equation and an analogy to the particle physics

2.1 BCS theory

Let us consider the BCS Hamiltonian:

$$H = \int d\mathbf{r} \sum_{\sigma} \psi_{\sigma}^{\dagger}(\mathbf{r}) \left(\frac{\mathbf{p}^2}{2m} - \mu \right) \psi_{\sigma}(\mathbf{r}) \\ -g \int d\mathbf{r} \psi_{\uparrow}^{\dagger}(\mathbf{r}) \psi_{\downarrow}^{\dagger}(\mathbf{r}) \psi_{\downarrow}(\mathbf{r}) \psi_{\uparrow}(\mathbf{r}), \tag{2}$$

where σ is the spin index $\uparrow$ and $\downarrow$, μ is the chemical potential and $g > 0$ is the coupling constant of the attractive interaction. In the momentum space, this is written as

$$H = \sum_{k\sigma} \xi_k c_{k\sigma}^{\dagger} c_{k\sigma} - g \frac{1}{V} \sum_{kk'q} c_{k'\uparrow}^{\dagger} c_{-k'+q\downarrow}^{\dagger} c_{-k+q\downarrow} c_{k\uparrow}, \tag{3}$$

where $\xi_k = \epsilon_k - \mu$ for the electron dispersion ϵ_k. The corresponding Lagrangian density is

$$\mathcal{L} = \sum_{\sigma} \psi_{\sigma}^{\dagger}(x) \left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 + \mu \right) \psi_{\sigma}(x) \\ +g \psi_{\uparrow}^{\dagger}(x) \psi_{\downarrow}^{\dagger}(x) \psi_{\downarrow}(x) \psi_{\uparrow}(x). \tag{4}$$

Using the Nambu notation[67],

$$\psi(x) = \begin{pmatrix} \psi_{\uparrow}(x) \\ \psi_{\downarrow}^{\dagger}(x) \end{pmatrix}, \tag{5}$$

the Lagrangian density becomes

$$\mathcal{L} = \psi^{\dagger} \left(\sigma_0 i\hbar \frac{\partial}{\partial t} - \sigma_3 \xi(\nabla) \right) \psi - \frac{g}{4} \left[(\psi^{\dagger}\psi)^2 - (\psi^{\dagger}\sigma_3\psi)^2 \right] \tag{6}$$

where σ_0 is the unit matrix and $\xi(\nabla) = -\hbar^2\nabla^2/(2m) - \mu = \mathbf{p}^2/(2m) - \mu$. The vacuum partition function is represented by a functional integral,

$$Z = \int d\psi^{\dagger} d\psi \exp\left(\frac{i}{\hbar} \int d^d x \mathcal{L} \right). \tag{7}$$

d is the space-time dimension. This can be written in a bilinear form by applying a Hubbard-Stratonovich transformation,

$$\exp\left(\frac{i}{\hbar} g \int d^d x \psi_{\uparrow}^{\dagger} \psi_{\downarrow}^{\dagger} \psi_{\downarrow} \psi_{\uparrow} \right) = \int d\Delta^* d\Delta \\ \exp\left[-\frac{i}{\hbar} \int d^d x \left(\Delta^* \psi_{\downarrow}\psi_{\uparrow} + \Delta \psi_{\uparrow}^{\dagger}\psi_{\downarrow}^{\dagger} + \frac{1}{g}|\Delta|^2 \right) \right], \tag{8}$$

where Δ^* and Δ are auxiliary fields and an overall normalization factor is excluded. The partition function has the form

$$Z = \int d\psi^{\dagger} d\psi \int d\Delta^* d\Delta \exp\left(\frac{i}{\hbar} \int d^d x \mathcal{L}_{eff} \right), \tag{9}$$

where

$$\mathcal{L}_{eff} = \psi^{\dagger} \left[\sigma_0 i\hbar \frac{\partial}{\partial t} - \sigma_3 \xi(\nabla) - \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix} \right] \psi - \frac{1}{g}|\Delta|^2. \tag{10}$$

The field equations obtained by variation of the Lagrangian are

$$\left[i\hbar \frac{\partial}{\partial t} - \sigma_3 \xi(\nabla) - \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix} \right] \psi = 0, \tag{11}$$

$$\Delta = g \psi_{\uparrow} \psi_{\downarrow}. \tag{12}$$

The equation for Δ shows that Δ describes a pair of electrons that forms a spin-singlet. It is clear that there is a similarity between this equation and the Dirac equation. If we approximate Δ by its average $\bar{\Delta} = g\langle \psi_{\uparrow}\psi_{\downarrow} \rangle$, we obtain a self-consistency equation for $\bar{\Delta}$. By performing the Grassmann integration over the fields $\psi^{\dagger}$ and ψ, we obtain the effective action

$$S(\Delta^*, \Delta) = -\frac{1}{g} \int d^d x |\Delta(x)|^2 - i\hbar \mathrm{Tr} \ln \begin{pmatrix} p_0 - \xi(\mathbf{p}) & -\Delta(x) \\ -\Delta^*(x) & p_0 + \xi(\mathbf{p}) \end{pmatrix}, \tag{13}$$

for which the partition function is

$$Z = \int d\Delta^* d\Delta \exp\left(\frac{i}{\hbar} S(\Delta^*, \Delta) \right). \tag{14}$$

Now the averaged value $\bar{\Delta}$ of the gap function Δ is determined by adopting the saddle point approximation. The field equation reads

$$\frac{\delta S(\bar{\Delta}^*, \bar{\Delta})}{\delta \bar{\Delta}^*} = 0. \tag{15}$$

We obtain a solution assuming that $\bar{\Delta} > 0$ is a constant. This yields

$$\frac{1}{g} \bar{\Delta} = i\hbar \mathrm{Tr} G_0(p) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{16}$$

where $G_0(p)$ is the Green function including $\bar{\Delta}$,

$$G_0(p) = \begin{pmatrix} p_0 - \xi(\mathbf{p}) & -\bar{\Delta} \\ -\bar{\Delta}^* & p_0 + \xi(\mathbf{p}) \end{pmatrix}^{-1} \\ = \frac{1}{p_0^2 - E(\mathbf{p})^2 + i\delta} \begin{pmatrix} p_0 + \xi(\mathbf{p}) & \bar{\Delta} \\ \bar{\Delta}^* & p_0 - \xi(\mathbf{p}) \end{pmatrix}. \tag{17}$$

Here,

$$E(\mathbf{p}) = \sqrt{\xi(\mathbf{p})^2 + \bar{\Delta}^2} \tag{18}$$

is the single-particle excitation energy. Then we obtain the gap equation

$$\frac{1}{g} = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{E(\mathbf{k})}. \tag{19}$$

The superconducting gap is

$$\bar{\Delta} = 2\hbar\omega_D \exp\left(-\frac{1}{\rho g} \right), \tag{20}$$

with the energy cutoff $\hbar\omega_D$ and the density of states ρ at the Fermi energy.

2.2 Nambu-Jona-Lasinio Model

The Nambu-Jona-Lasinio model is

$$\mathcal{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi + g[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2], \tag{21}$$

which has the form similar to the BCS model. We set $\hbar = 1$ in this section. γ_μ and γ_5 are Dirac gamma matrices. This Lagrangian is invariant under the particle number and chiral transformations,

$$\psi \to \exp(i\alpha)\psi, \quad \bar{\psi} \to \bar{\psi}\exp(-i\alpha) \tag{22}$$

$$\psi \to \exp(i\gamma_5\alpha)\psi, \quad \bar{\psi} \to \bar{\psi}\exp(i\gamma_5\alpha). \tag{23}$$

In a similar way after the spontaneous symmetry breaking, the fermion (nucleon) acquires a mass $m \propto 2g\langle\bar{\psi}\psi\rangle$.

Using an identity

$$1 = \text{const.} \int d\sigma' d\pi' \exp i \int d^4x \Big[-\frac{1}{4g}(\sigma'^2 + \pi'^2) \Big], \tag{24}$$

the partition function is written as

$$Z = \int d\bar{\psi} d\psi d\sigma' d\pi' \exp i \int d^4x \Big[\bar{\psi} i\gamma^\mu \partial_\mu \psi + g(\bar{\psi}\psi)^2 - g(\bar{\psi}\gamma_5\psi)^2 - \frac{1}{4g}(\sigma'^2 + \pi'^2) \Big]. \tag{25}$$

We define new σ and π fields by

$$\sigma' = \sigma + 2g\bar{\psi}\psi, \tag{26}$$

$$\pi' = \pi + 2gi\bar{\psi}\gamma_5\psi, \tag{27}$$

then we have

$$Z = \int \bar{\psi} d\psi d\sigma d\pi \exp\left(i \int d^4x \mathcal{L}_{eff} \right), \tag{28}$$

where

$$\begin{aligned} \mathcal{L}_{eff} &= \mathcal{L} - \frac{1}{4g}[(\sigma + 2g\bar{\psi}\psi)^2 + (\pi + 2gi\bar{\psi}\gamma_5\psi)^2] \\ &= \bar{\psi}[i\gamma^\mu\partial_\mu - (\sigma + i\gamma_5\pi)]\psi - \frac{1}{4g}(\sigma^2 + \pi^2). \end{aligned} \tag{29}$$

Then we obtain the effective action

$$\begin{aligned} S_{eff}(\sigma, \pi) &= -i \ln \det[i\gamma^\mu\partial_\mu - (\sigma + i\gamma_5\pi)] \\ &\quad -\frac{1}{4g}\int d^4x(\sigma^2 + \pi^2) \\ &= -i\text{Tr}\ln[i\gamma^\mu\partial_\mu - (\sigma + i\gamma_5\pi)] \\ &\quad -\frac{1}{4g}\int d^4x(\sigma^2 + \pi^2). \end{aligned} \tag{30}$$

The saddle point approximation leads to a solution such that $\sigma = \sigma_0$ is a constant and $\pi = 0$. The equation for σ_0 is

$$\sigma_0 = 2ig\text{Tr}\frac{1}{i\gamma^\mu\partial_\mu - \sigma_0}. \tag{31}$$

Because σ_0 is the mass m of the fermion ψ, the mass m is determined by

$$1 = 8gi \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2} = 8g \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2}, \tag{32}$$

which has a nontrivial solution $m \neq 0$ when

$$0 < \frac{2\pi^2}{g\Lambda^2} < 1. \tag{33}$$

In this case the mass m is determined by

$$\frac{2\pi^2}{g\Lambda^2} = 1 - \frac{m^2}{\Lambda^2}\ln\left(1 + \frac{\Lambda^2}{m^2}\right). \tag{34}$$

We define the field h_σ by

$$\sigma = \sigma_0 + h_\sigma, \tag{35}$$

then h_σ is a massive boson whose mass is $2m$. The field π represents a massless boson which is the Nambu-Goldstone boson. The result that the field h_σ acquires the mass $2m$ is well understood by an analogy to superconductivity, where the excitation energy is $2\bar{\Delta}$ when a pair of electrons is excited above the Fermi energy. The model has been generalized to a more realistic two-flavor model:

$$\mathcal{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi + g[(\bar{\psi}\psi)^2 - \sum_i (\bar{\psi}\gamma_5\tau_i\psi)(\bar{\psi}\gamma_5\tau_i\psi)]. \tag{36}$$

As is obvious from the discussion here, the nucleon mass generation is very analogous to the gap generation in superconductors. We note that the mass is finite only when $0 < 2\pi^2/g\Lambda^2 < 1$ holds in the Nambu-Jona-Lasinio model while the superconducting gap always exists as far as $g > 0$.

3 Multi-gap superconductivity

We give a brief survey on the research of multi-gap superconductivity[69]. Two years after the BCS theory was proposed[5], an extension to two overlapping bands was considered by Moskalenko[1] and Suhl, Matthias and Walker[2]. After these works, Peretti[3], Kondo[4] and Geilikman[70] reconsidered superconductors with multiple bands. The motivation of Kondo's work is to understand the small isotope effect observed for some transition metal superconductors. Kondo investigated the exchange-like integral between different bands, which is a non-phonon effective attractive interaction, and proposed a possibility of small, being less than 0.5, or vanishing of the isotope effect of the critical temperature T_c using the two-band model. It was found by early works that the critical temperature is enhanced higher than both of critical temperatures of uncoupled superconductors due to the interband coupling. The critical field $H_{c2}(0)$ and the sizable positive curvature of $H_{c2}(T)$ in YNi_2B_2C and $LuNi_2B_2C$ were analyzed on the basis of an effective two-band model. The Ginzburg-Landau model was extended to include two conduction bands[42, 71–73]. Kondo, at the same time, introduced different phases assigned to two different gaps with phase difference π. This indicates that we can take the phase difference φ to be 0 or π for the two-band model. A simple generalization to a three-band model was investigated much later than Kondo's work. It was shown independently[38–40] that the phase difference other than 0 or π is possible. It was indicated that the intermediate value of the phase difference φ leads to time reversal symmetry breaking, which is a new state in three-band superconductors. There have been many works for a pairing state with time reversal symmetry breaking[41–48, 51, 52, 57, 74–76] with relation to iron-based superconductors[77], and also from the viewpoint of holographic superconductors[78–80].

Leggett[63] considered small fluctuation of phase difference, which yields fluctuation in the density of Cooper pairs. This indicates a possibility of a collective excitation of phase difference mode. Leggett examined the Josephson term $-J\cos(\varphi)$ using the expansion $\cos(\varphi) = 1-(1/2)\varphi^2+\cdots$. In the presence of large fluctuation of φ we are not allowed to use this approximation. In this situation we must employ a sine-Gordon model. This model has a kink solution[81] with fluctuation from $\varphi = 0$ to 2π, which results in a new collective mode[60, 82–86].

An intensive study of multi-gap superconductivity started since the discovery of MgB_2, and especially iron-based superconductors. A new kind of superconductivity, called the type 1.5, was proposed for MgB_2[65] where it seems that there is an attractive inter-vortex interaction preventing the formation of Abrikosov vortex lattice. A theoretical prediction was given based on the model with vanishing Josephson coupling[87]. There are some controversial on this subject[88–90]. We expect that the Higgs mode plays a role in this issue because Higgs mode will produce an attractive force between vortices. A three-band model is now considered as a model for iron-based superconductors and the time reversal symmetry breaking is investigated intensively.

4 Effective action of multi-gap superconductors

Let us consider the Hamiltonian for multi-gap superconductors:

$$H = \sum_{i\sigma}\int d\mathbf{r}\psi^\dagger_{i\sigma}(\mathbf{r})K_i(\mathbf{r})\psi_{i\sigma}(\mathbf{r}) - \sum_{ij} g_{ij}\int d\mathbf{r}\psi^\dagger_{i\uparrow}(\mathbf{r})\psi^\dagger_{i\downarrow}(\mathbf{r})\psi_{j\downarrow}(\mathbf{r})\psi_{j\uparrow}(\mathbf{r}), \tag{37}$$

where i and j (=1,2,$\cdots$) are band indices. $K_i(\mathbf{r})$ stands for the kinetic operator: $K_i(\mathbf{r}) = p^2/(2m_i) - \mu \equiv \xi_i(\mathbf{p})$ where μ is the chemical potential. We assume that $g_{ij} = g^*_{ji}$. The second term indicates the pairing interaction with the coupling constants g_{ij}. This model is a simplified version of multi-band model where the coupling constants g_{ij} are assumed to be constants.

In the functional-integral formulation, using the Hubbard-Stratonovich transformation, the partition function is expressed as follows:

$$Z = \int d\psi_\uparrow d\psi_\downarrow \int d\Delta^* d\Delta \exp\left(-\int_0^\beta d\tau d^d x \sum_{ij}\Delta^*_i (G^{-1})_{ij}\Delta_j\right) \times \exp\Big[-\sum_j \int d\tau d^d x (\psi^*_{j\uparrow}\psi_{j\downarrow})\begin{pmatrix}\partial_\tau + \xi_j(\mathbf{p}) & \Delta_j \\ \Delta^*_j & \partial_\tau - \xi_j(\mathbf{p})\end{pmatrix}\begin{pmatrix}\psi_{j\uparrow}\\ \psi^*_{j\downarrow}\end{pmatrix}\Big], \tag{38}$$

where $G = (g_{ij})$ is the matrix of coupling constants. $(G^{-1})_{ij}$ $(i \neq j)$ indicates the Josephson coupling. The condition for the matrix G has been discussed in Ref.[91].

In order to obtain the effective action for phase variables θ_j, we perform the gauge transformation

$$\begin{pmatrix}\psi_{j\uparrow}\\ \psi^*_{j\downarrow}\end{pmatrix} \to \begin{pmatrix} e^{i\theta_j}\psi_{j\uparrow}\\ e^{-i\theta_j}\psi^*_{j\downarrow}\end{pmatrix}, \tag{39}$$

so that Δ_j are real and positive. The effective action is written in the form

$$S = \sum_{ij}\int d\tau d^d x \Delta_i (G^{-1})_{ij}\Delta_j \cos(2(\theta_i - \theta_j)) - \mathrm{Tr}\ln\begin{pmatrix}\partial_\tau + i\partial_\tau\theta_j + \xi_j(\mathbf{p}+\nabla\theta_j) & \Delta_j \\ \Delta_j & \partial_\tau + i\partial_\tau\theta_j - \xi_j(\mathbf{p}+\nabla\theta_j)\end{pmatrix}. \tag{40}$$

We define the fluctuation mode (Higgs mode) h_j of the amplitude of Δ_j as

$$\Delta_j = \bar{\Delta}_j + h_j, \tag{41}$$

where $\bar{\Delta}_j$ is the gap function given by the saddle point approximation. We define

$$\mathbf{a}_j = \nabla\theta_h, \quad a_{0j} = i\partial_\tau\theta_j. \tag{42}$$

Then, the effective action is written in the form:

$$S = \sum_{ij}\int d\tau d^d x \Delta_i (G^{-1})_{ij}\Delta_j \cos(2(\theta_i - \theta_j)) - \sum_j \mathrm{Tr}\ln\left(S^{-1}_{Fj} + \begin{pmatrix}0 & h_j\\ h_j & 0\end{pmatrix} + V_j\right), \tag{43}$$

where

$$V_j = \frac{1}{2m_j}(\mathbf{p}\cdot\mathbf{a}_j + \mathbf{a}_j\cdot\mathbf{p}) + \left(a_{0j} + \frac{1}{2m_j}\mathbf{a}^2_j\right)\sigma_3, \tag{44}$$

and S_{Fj} is defined by

$$S^{-1}_{Fj}(i\omega_n, \mathbf{p}) = \begin{pmatrix}-i\omega_n + \xi_j(\mathbf{p}) & \bar{\Delta}_j \\ \bar{\Delta}_j & -i\omega_n - \xi_j(\mathbf{p})\end{pmatrix}, \tag{45}$$

n the momentum space where ω_n is the Matsubara frequency. G is the matrix of coupling constants g_{ij}: $G = (g_{ij})$. This action is expanded in the form:

$$S = \sum_{ij}\int d\tau d^d x \Delta_i (G^{-1})_{ij}\Delta_j \cos(2(\theta_i-\theta_j)) \\ -\sum_j \mathrm{Tr}\ln S_{Fj}^{-1} + \sum_j \mathrm{Tr}\sum_{\ell=1}^{\infty}\frac{1}{\ell}(-1)^\ell \Big[S_{Fj}\begin{pmatrix} o & h_j \\ h_j & 0\end{pmatrix} \\ +S_{Fj}V_j\Big]^\ell. \quad (46)$$

5 Nambu-Goldstone and Leggett modes

5.1 Effective action

The effective action for phase modes θ_j is given by the usual quadratic form with the Josephson coupling. The lowest-order contribution is

$$S^{(\ell=1)}[\theta] = \sum_j \mathrm{Tr} n_j\left(i\partial_\tau\theta_j + \frac{1}{2m_j}(\nabla\theta_j)^2\right), \quad (47)$$

where

$$n_j = \int\frac{d^dk}{(2\pi)^d}\Big[1-\frac{\xi_j(\mathbf{k})}{E_j(\mathbf{k})}(1-2f(E_j))\Big], \quad (48)$$

with $E_j = \sqrt{\xi_j^2 + \bar{\Delta}_j^2}$. The second term with $\ell = 2$ is

$$S^{(\ell=2)}[\theta] = \frac{1}{2}\sum_j \mathrm{Tr} S_{Fj}\sigma_3 S_{Fj}\sigma_3\left(i\partial_\tau\theta_j + \frac{1}{2m_j}(\nabla\theta_j)^2\right)^2. \quad (49)$$

Then the quadratic terms of the Nambu-Goldstone-Leggett modes is given by

$$S^{(2)}[\theta] = \sum_j\int d\tau d^dx\Big[\rho_j(\partial_\tau\theta_j)^2 + n_j\frac{1}{2m_j}(\nabla\theta_j)^2\Big] \\ +\sum_{ij}\int d\tau d^dx \bar{\Delta}_i(G^{-1})_{ij}\bar{\Delta}_j\cos(2(\theta_i-\theta_j)), \quad (50)$$

where ρ_j is the density of states in the j-th band. The Nambu-Goldstone-Leggett modes become massive due to the Josephson term. The gap of the Leggett mode (phase-difference mode) is determined by Josephson couplings. In general, the dynamics of the Leggett mode is described by the sine-Gordon model[42, 82]. A generalization of the sine-Gordon model has also been discussed recently[92].

5.2 Nambu-Goldstone-Leggett mode for neutral superconductors

Let us consider a two-band neutral superconductor, where the action density reads

$$\mathcal{L}_E[\theta] = \rho_1(\partial_\tau\theta_1)^2 + \rho_2(\partial_\tau\theta_2)^2 + \frac{n_1}{2m_1}(\nabla\theta_1)^2 \\ +\frac{n_2}{2m_2}(\nabla\theta_2)^2 + 2\gamma_{12}\bar{\Delta}_1\bar{\Delta}_2\cos(2(\theta_1-\theta_2)). \quad (51)$$

Here $\gamma_{ij} = (G^{-1})_{ij}$. We assume that γ_{12} is negative and θ_i are small, so that we expand the potential $\cos(2(\theta_1 - \theta_2))$ in terms of $\theta_1 - \theta_2$. The dispersion relations of the Nambu-Goldstone mode and the Leggett mode are determined by[98, 101]:

$$\det\begin{pmatrix} \rho_1\omega^2 - \frac{n_1}{2m_1}k^2 - 4|\gamma_{12}|\bar{\Delta}_1\bar{\Delta}_2 & 4|\gamma_{12}|\bar{\Delta}_1\bar{\Delta}_2 \\ 4|\gamma_{12}|\bar{\Delta}_1\bar{\Delta}_2 & \rho_2\omega^2 - \frac{n_2}{2m_2}k^2 - 4|\gamma_{12}|\bar{\Delta}_1\bar{\Delta}_2\end{pmatrix} \\ = 0, \quad (52)$$

where we performed an analytic continuation $i\omega_n \to \omega$. The dispersion relations of the Nambu-Goldstone and Leggett modes are, respectively, given by

$$\omega^2 = \frac{1}{\rho_1+\rho_2}\left(\frac{n_1}{2m_1}+\frac{n_2}{2m_2}\right)k^2 = v_N^2k^2, \quad (53)$$

$$\omega^2 = 4\frac{\rho_1+\rho_2}{\rho_1\rho_2}|\gamma_{12}|\bar{\Delta}_1\bar{\Delta}_2 + \frac{1}{\rho_1+\rho_2}\left(\frac{n_1\rho_2}{2m_1\rho_1}+\frac{n_2\rho_1}{2m_2\rho_2}\right)k^2 \\ = \omega_J^2 + v_L^2k^2, \quad (54)$$

where

$$v_N^2 = \frac{1}{3}\frac{\rho_1v_{F1}^2+\rho_2v_{F2}^2}{\rho_1+\rho_2}, \quad (55)$$

$$v_L^2 = \frac{1}{3}\frac{\rho_2v_{F1}^2+\rho_1v_{F2}^2}{\rho_1+\rho_2}, \quad (56)$$

$$\omega_J^2 = 4\frac{\rho_1+\rho_2}{\rho_1\rho_2}|\gamma_{12}|\bar{\Delta}_1\bar{\Delta}_2. \quad (57)$$

v_{Fj} is the Fermi velocity of the j-th band.

5.3 Plasma and Leggett modes

For charged superconductors, we introduce the scalar potential Φ. One mode of the Nambu-Goldstone modes becomes a massive plasma mode in the presence of the Coulomb potential Φ. Let us consider the action density given as

$$\mathcal{L}_E[\theta] = \sum_j\left[\rho_j(\partial_\tau\theta_j - e\Phi)^2 + n_j\frac{1}{2m_j}(\nabla\theta_j)^2\right] + \frac{1}{8\pi}(\nabla\Phi)^2 \\ +\sum_{ij}\bar{\Delta}_i(G^{-1})_{ij}\bar{\Delta}_j\cos(2(\theta_i-\theta_j)), \quad (58)$$

where e is the charge of the electron. We integrate out the field Φ to obtain the effective action for the fields

θ_j:

$$\mathcal{L}_E[\theta] = \frac{1}{8\pi e^2\rho(0)^2}\sum_{jj'a}\rho_j\rho_{j'}\partial_\tau\zeta_{ja}\partial_\tau\zeta_{j'a} + \sum_{ja}\frac{n_j}{2m_j}\zeta_{ja}^2 + \sum_j \rho_j\left(\partial_\tau\theta_j\right)^2 - \frac{1}{\rho(0)}\left(\sum_j \rho_j\partial_\tau\theta_j\right)^2 + \sum_{ij}\bar{\Delta}_i(G^{-1})_{ij}\bar{\Delta}_j\cos(2(\theta_i-\theta_j)) + \cdots, \tag{59}$$

where we put $\zeta_{ja} = \nabla_a\theta_j$ and $\rho(0) = \sum_j \rho_j$, and the index a takes x, y and z. ζ_{ja} represents the massive mode called the plasma mode. The Nambu-Goldstone mode was absorbed by the Coulomb potential to be the massive plasma mode. There are three plasma modes ζ_{ja} for $a = x$, y and z for each band.

In the single-band case, the plasma frequency is

$$\omega_{pl,a}^2 = 4\pi e^2 n/m_a, \tag{60}$$

where n is the electron density $n = n_{j=1}$. In the case with large anisotropy such as $m_z \gg m_x, m_y$, the one mode ζ_z has a small plasma frequency.

In the two-band model with equivalent bands, i.e., $\xi_1 = \xi_2 = p^2/(2m) - \mu$ for simplicity, the Lagrangian reads

$$\begin{aligned}\mathcal{L}_E &= \rho_F(\partial_\tau\theta_1 - e\Phi)^2 + \rho_F(\partial_\tau\theta_2 - e\Phi)^2 \\ &\quad + \frac{n}{2m}\left((\nabla\theta_1)^2 + (\nabla\theta_2)^2\right) + \frac{1}{8\pi}(\nabla\Phi)^2 \\ &\quad +2\gamma\bar{\Delta}_1\bar{\Delta}_2\cos(2(\theta_1-\theta_2)) + \cdots \\ &= 2\rho_F\left(\frac{1}{2}\partial_\tau\phi - e\Phi\right)^2 + \frac{n}{4m}(\nabla\phi)^2 + \frac{1}{8\pi}(\nabla\Phi)^2 \\ &\quad +\frac{1}{2}\rho_F(\partial_\tau\varphi)^2 + \frac{n}{4m}(\nabla\varphi)^2 + 2\gamma\bar{\Delta}_1\bar{\Delta}_2\cos(2\varphi) + \cdots,\end{aligned} \tag{61}$$

where $\cdots$ indicates higher order terms including the coupling terms between amplitude modes and phase modes. We introduced the scalar potential Φ which represents the Coulomb interaction and defined

$$\phi = \theta_1 + \theta_2, \quad \varphi = \theta_1 - \theta_2. \tag{62}$$

γ denotes the Josephson coupling strength given by $\gamma = \gamma_{12} \equiv (G^{-1})_{12}$. ρ_F is the density of states at the Fermi level. The derivative of the total phase $\nabla\phi$ represents the plasma mode with the plasma frequency $\omega_p^2 = 4\pi ne^2/m$. This is seen by writing the terms of ϕ in the following form by integrating out the scalar potential Φ:

$$\frac{1}{2}\rho_F\left(\frac{\omega_n^2}{\mathbf{k}^2 + 16\pi\rho_F e^2} + \frac{n}{2m\rho_F}\right)\mathbf{k}^2|\phi(i\omega_n,\mathbf{k})|^2, \tag{63}$$

after the Fourier transformation. This indicates that the plasma mode is described by the derivative of the total phase $\nabla\phi$. By performing the analytic continuation $i\omega_n \to \omega + i\delta$, we obtain the dispersion relation as

$$\omega^2 = \omega_{pl}^2 + c_s^2\mathbf{k}^2, \tag{64}$$

where $\omega_{pl}^2 = 8\pi ne^2/m$ and $c_s^2 = n/(2m\rho_F)$. ω_{pl}^2 for the two-band model is twice that of the single-band model.

The Lagrangian of the phase difference mode (Leggett mode) φ is given by the sine-Gordon model. This mode is a massless mode if the Josephson coupling γ vanishes. When φ is small, the sine-Gordon model describes an oscillation mode, by using $\cos\varphi = 1 - \varphi^2/2 + \cdots$. We assume that γ is positive so that φ describes a stable oscillation mode. The frequency of this mode is proportional to the gap amplitude:

$$\omega_J = 2\sqrt{\frac{2|\gamma|}{\rho_F}}\bar{\Delta}. \tag{65}$$

The dispersion relation is given as

$$\omega^2 = \omega_J^2 + \frac{1}{3}v_F^2\mathbf{k}^2. \tag{66}$$

In the general case where the two bands are not equivalent, the dispersion of the Leggett mode is given by

$$\begin{aligned}\omega^2 &= \omega_J^2 + \frac{\rho_1+\rho_2}{\rho_1\rho_2}\frac{1}{n_{s1}/2m_1 + n_{s2}/2m_2}\frac{n_{s1}}{2m_1}\frac{n_{s2}}{2m_2}k^2 \\ &= \omega_J^2 + \frac{1}{9}\frac{1}{v_N^2}v_{F1}^2 v_{F2}^2 k^2.\end{aligned} \tag{67}$$

This kind of oscillation mode is known as the Josephson plasma mode[93–97]. In MgB_2 the frequency of the oscillation mode (Leggett mode) was estimated to be 1.6 or 2THz[98]. There are two superconducting gaps in MgB_2; their magnitudes are given by $\Delta_1 \simeq 1.2$meV − 3.7meV (π band, smaller gap) and $\Delta_2 \simeq 6.4$meV − 6.8meV (σ band, larger gap)[99]. Thus, the frequency of the Leggett mode is larger than $2\Delta_1$. The observation of the Leggett mode in MgB_2 was recently reported by Raman scattering measurements[100].

In an N-gap superconductor, the plasma frequency is given by the formula

$$\omega_{pl,a}^2 = 4\pi e^2\frac{n_1\cdots n_N}{m_{1a}\cdots m_{Na}} \times \frac{\rho(0)^2}{\rho_1^2\frac{n_2\cdots n_N}{m_{2a}\cdots m_{Na}} + \cdots + \rho_N^2\frac{n_1\cdots n_{N-1}}{m_{1a}\cdots m_{N-1,a}}}. \tag{68}$$

When N gaps are equivalent, this formula reduces to

$$\omega_{pl,a}^2 = 4\pi e^2\frac{n}{m_a}N. \tag{69}$$

When one conduction band has an effective heavy mass compared to other bands, the plasma frequency is determined by its heavy mass.

In general, in an N-gap superconductor, there are $N-1$ Leggett modes because one mode becomes a massive mode with the plasma frequency by coupling to the Coulomb potential. When N bands are equivalent, the Josephson term is invariant under an S_N group action. When there is an anisotropy that breaks the equivalence among several bands, we have lower symmetry than S_N.

6 Higgs mode

Let us discuss the fluctuation of the amplitude of gap functions, which is called the Higgs mode. Recently, there has been an increasing interest in a role of the Higgs mode in superconductors[102–106]. In the relativistic model considered by Nambu and Jona-Lasinio, the Higgs mass is just twice the magnitude of superconducting gap $\bar{\Delta}$[107, 108]. This results in the mass ratio given as[109]

$$m_{NG} : \bar{\Delta} : m_H = 0 : 1 : 2 \tag{70}$$

where m_{NG} is the mass of the Nambu-Goldstone boson and m_H is that of the Higgs boson.

The action up to the second order of h is

$$S^{(2)}[h] = \frac{1}{2}\sum_j \mathrm{Tr}\Big[S_{Fj}\sigma_1 h_j S_{Fj}\sigma_1 h_j\Big] + \sum_{ij}\int d\tau d^d x h_i (G^{-1})_{ij} h_j \cos(2(\theta_i - \theta_j)). \tag{71}$$

6.1 Effective action near T_c

When the temperature T is near T_c, the effective action is given by the time-dependent Ginzburg-Landau (TDGL) action. For the single-band case, we have

$$S^{(2)}[h] = \int d\tau d^d x\rho\Big[\frac{\pi}{8k_BT_c} hi\frac{\partial h}{\partial \tau} + \frac{1}{4\epsilon_F}\ln\left(\frac{2e^\gamma\omega_c}{\pi k_B T_c}\right)h\frac{\partial h}{\partial\tau} + \frac{7\zeta(3)}{48\pi^2}\frac{v_F^2}{(k_BT_c)^2}(\nabla h)^2 + \frac{7\zeta(3)}{4\pi^2(k_BT_c)^2}\bar{\Delta}^2 h^2\Big]. \tag{72}$$

The time dependence gives the dissipation effect so that the Higgs mode may not be defined in this region.

6.2 Effective action at low temperature

At low temperature, in contrast, the Higgs mode can be defined clearly. We employ the approximation that the density of states is constant. Then, the action in the single-band case is

$$S^{(2)}[h] = \int d\tau d^d x\rho\Big[\frac{1}{12\bar{\Delta}^2}\left(\frac{\partial h}{\partial\tau}\right)^2 + \frac{v_F^2}{36\bar{\Delta}^2}(\nabla h)^2 + h^2\Big]. \tag{73}$$

This is obtained by evaluating the Higgs boson one-loop contribution given by

$$\Pi(\mathbf{q}, i\epsilon) = \frac{1}{\beta}\sum_n \frac{1}{V}\sum_{\mathbf{p}} \mathrm{tr}\Big[S_F(\mathbf{p}+\mathbf{q}, i\omega_n + i\epsilon)\sigma_1 \times S_F(\mathbf{p}, i\omega_n)\sigma_1\Big], \tag{74}$$

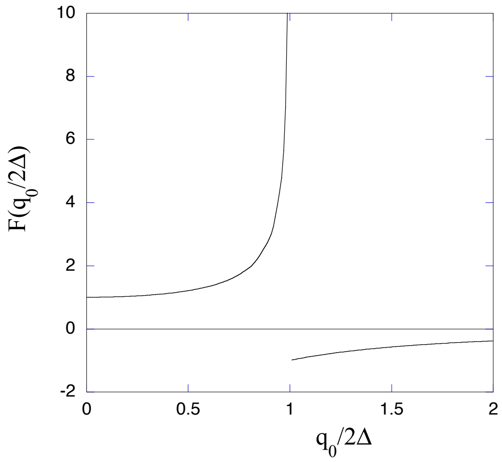

Fig. 1 Re$F(x)$ as a function of $x = q_0/2\Delta$. $F(x)$ has a singularity at $x = 1$.

where we neglected the terms with higher-order derivatives. At absolute zero, $\Pi(\mathbf{q} = 0, q_0)$ (where $q_0 = i\epsilon$) is calculated as[110]

$$\Pi(\mathbf{q} = 0, q_0) = -\rho(0)\int d\xi \frac{1}{E(\xi)} + 2\rho(0)\Big[1 - \left(\frac{q_0}{2\bar{\Delta}}\right)^2\Big]F\left(\frac{q_0}{2\bar{\Delta}}\right), \tag{75}$$

where $E(\xi) = \sqrt{\xi^2 + \bar{\Delta}^2}$. The second term in the action $S^{(2)}[h]$ in eq.(71) cancels the first term of $\Pi(\mathbf{q} = 0, q_0)$ by the gap equation. When $q_0/2\bar{\Delta} < 1$, $F(q_0/2\bar{\Delta})$ is given by

$$F(x) = \frac{1}{x\sqrt{1-x^2}}\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right), \tag{76}$$

for $x < 1$. When $q_0/2\bar{\Delta} > 1$, we obtain by the analytic continuation

$$F(x) = \frac{1}{2x\sqrt{x^2-1}}\log\Big|\frac{x - \sqrt{x^2-1}}{x + \sqrt{x^2-1}}\Big| + i\frac{\pi}{2x\sqrt{x^2-1}}, \tag{77}$$

for $x > 1$. The real part of $F(x)$ is shown in Fig.1, and the behavior of $\Pi(\mathbf{q} = 0, q_0)$ is shown for $q_0 < 2\bar{\Delta}$ in Fig.2. Then, for small $q_0/2\bar{\Delta}$, we have

$$\frac{1}{g} + \frac{1}{2}\Pi(\mathbf{q} = 0, q_0) = \rho\Big[1 - \frac{1}{3}\left(\frac{q_0}{2\bar{\Delta}}\right)^2 + \dots\Big], \tag{78}$$

where $g = g_{11}$ This results in eq.(73). The quantity $1/g + (1/2)\Pi(\mathbf{q} = 0, q_0)$ has a zero at

$$q_0 = 2\bar{\Delta}. \tag{79}$$

In the relativistic model, the same calculation leads to $1 - (q_0/2\bar{\Delta})^2$. This gives the mass $m_H = 2\bar{\Delta}$.

In the multi-band case, the action is given by the quadratic form,

$$S^{(2)}[h] = \int d\tau d^d x \sum_{j\ell}\eta_j H_{j\ell}\eta_\ell. \tag{80}$$

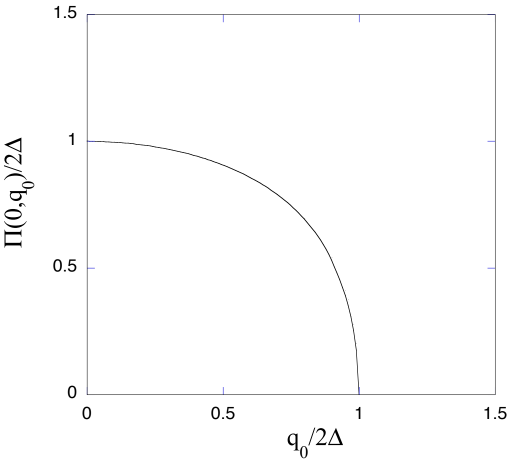

Fig. 2 $\Pi(\mathbf{q}=0,q_0)$ as a function of $x=q_0/2\Delta$. The constant which is canceled by $1/g$ is neglected.

The excitation spectra is determined from the condition $\det H=0$. At low temperatures the spectrum has a gap being proportional to the mean-field gap amplitude. For $N=2$ (two-band superconductor), the matrix $H_{j\ell}$ is written as

$$\begin{pmatrix} \gamma_{11}+\frac{1}{2}\Pi_1 & \gamma_{12} \\ \gamma_{21} & \gamma_{22}+\frac{1}{2}\Pi_2 \end{pmatrix}, \tag{81}$$

where Π_ℓ is

$$\Pi_\ell(\mathbf{q},i\epsilon)=\frac{1}{\beta}\sum_n\frac{1}{V}\sum_{\mathbf{p}}\mathrm{tr}\Big[S_{F\ell}(\mathbf{p}+\mathbf{q},i\omega_n+i\epsilon)\sigma_1 \times S_{F\ell}(\mathbf{p},i\omega_n)\sigma_1\Big]. \tag{82}$$

Then the dispersion relation of the Higgs mode $\omega=\omega(\mathbf{q})$ is given by a solution of the equation

$$1+\frac{1}{2}g_{11}\Pi_1(\mathbf{q},\omega)+\frac{1}{2}g_{22}\Pi_2(\mathbf{q},\omega) +\frac{1}{4}\det G\cdot\Pi_1(\mathbf{q},\omega)\Pi_2(\mathbf{q},\omega)=0, \tag{83}$$

where $\det G=g_{11}g_{22}-g_{12}g_{21}$.

In the multi-gap case, the action becomes

$$S^{(2)}[h]=\int d\tau d^dx\sum_j\Big[h_i(-\rho_jf_j+\rho_j\nu_j)h_j +\frac{1}{12\bar{\Delta}_j^{\,2}}\left(\frac{\partial h_j}{\partial\tau}\right)^2+\frac{v_{Fj}^2}{36\bar{\Delta}_j^{\,2}}(\nabla h_j)^2\Big] +\int d\tau d^dx\sum_{ij}h_i(G^{-1})_{ij}h_j, \tag{84}$$

where we defined

$$\rho_jf_j=\int\frac{d^dk}{(2\pi)^d}\frac{1}{2E_j}\left(1-2f(E_j)\right), \tag{85}$$

$$\rho_j\nu_j=\int\frac{d^dk}{(2\pi)^d}\frac{1}{\beta}\sum_n\frac{2\bar{\Delta}_j^{\,2}}{(\omega_n^2+\xi_j^2+\bar{\Delta}_j^{\,2})^2}. \tag{86}$$

We set the phase variables $\{\theta_j\}$ to take their equilibrium values that are assumed to be zero here. The excitation gaps of Higgs modes in the multi-gap model will be obtained by diagonalizing the Higgs kinetic terms. Since the h squared term in eq.(84) is closely related to the critical field H_{c2}, because this term is proportional to $1/\xi^2$ where ξ is the coherence length, we expect that the large upper critical field $H_{c2}(0)$ observed in iron-based superconductors[111] is understood by means of a multi-band model of Higgs fields.

7 Time-Reversal Symmetry Breaking

The gap function, defined as $\Delta_i(\mathbf{r})=-\sum_j g_{ij}\langle\psi_{j\downarrow}(\mathbf{r})\psi_{j\uparrow}(\mathbf{r})\rangle$, satisfies the gap equation

$$\Delta_i=\sum_j g_{ij}N_j\Delta_j\int d\xi_j\frac{1}{E_j}\tanh\left(\frac{E_j}{2k_BT}\right), \tag{87}$$

where N_j is the density of states at the Fermi surface in the j-th band and $E_j=\sqrt{\xi_j^2+|\Delta_j|^2}$. Δ_i in this section is the mean-field solution in section III which is obtained by a saddle-point approximation. We set

$$\zeta_j=\int_0^{\omega_{Dj}}d\xi_j\frac{1}{E_j}\tanh\left(\frac{E_j}{2k_BT}\right), \tag{88}$$

and $\gamma_{ij}=(G^{-1})_{ij}$ where $G=(g_{ij})$. We write the gap equation in the following form,

$$\begin{pmatrix} \gamma_{11}-N_1\zeta_1 & \gamma_{12} & \gamma_{13} & \cdots \\ \gamma_{21} & \gamma_{22}-N_2\zeta_2 & \gamma_{23} & \cdots \\ \gamma_{31} & \gamma_{32} & \gamma_{33}-N_3\zeta_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix}\Delta_1\\ \Delta_2\\ \Delta_3\\ \cdots\end{pmatrix}=0. \tag{89}$$

γ_{ij} $(i\neq j)$ gives the interband Josephson coupling between bands i and j[42].

When the gap functions Δ_j are complex-valued functions, the time-reversal symmetry is broken. The condition for TRSB is that the following equation for the imaginary part $\mathrm{Im}\Delta_j$ has a nontrivial solution:

$$\begin{pmatrix} \gamma_{12} & \gamma_{13} & \cdots \\ \gamma_{22}-N_2\zeta_2 & \gamma_{23} & \cdots \\ \gamma_{32} & \gamma_{33}-N_3\zeta_3 & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix}\mathrm{Im}\Delta_2\\ \mathrm{Im}\Delta_3\\ \cdots\end{pmatrix}=0, \tag{90}$$

where we adopt that Δ_1 is real for simplicity and γ_{ij} are real. We assume that $\gamma_{ij}=\gamma_{ji}$. In the case of $N=3$, the condition for TRSB has been obtained[43,47]. We have a necessary condition $\gamma_{12}\gamma_{23}\gamma_{13}>0$[39,40]. The determinant of each 2×2 matrix in eq.(90) should vanish so that non-trivial solution $\mathrm{Im}\Delta_j$ $(j=2,3)$ exist. Then we have

$$\gamma_{12}\gamma_{23}-(\gamma_{22}-N_2\zeta_2)\gamma_{13}=0, \tag{91}$$

$$(\gamma_{22}-N_2\zeta_2)(\gamma_{33}-N_3\zeta_3)-\gamma_{23}^2=0, \tag{92}$$

$$\gamma_{12}(\gamma_{33}-N_3\zeta_3)-\gamma_{12}\gamma_{23}=0. \tag{93}$$

When we assume $\gamma_{13} \neq 0$, we obtain

$$\gamma_{22} - N_2\zeta_2 = \gamma_{12}\gamma_{23}/\gamma_{13}. \tag{94}$$

Similarly, we have by assuming $\gamma_{12} \neq 0$

$$\gamma_{33} - N_3\zeta_3 = \gamma_{23}\gamma_{13}/\gamma_{12}. \tag{95}$$

From the gap equation $\gamma_{21}\Delta_1 + (\gamma_{22} - N_2\zeta_2)\Delta_2 + \gamma_{23}\Delta_3 = 0$, we obtain the relation

$$\frac{\Delta_1}{\gamma_{23}} + \frac{\Delta_2}{\gamma_{31}} + \frac{\Delta_3}{\gamma_{12}} = 0. \tag{96}$$

The complex numbers $\Delta_1/\gamma_{23}, \cdots$ form a triangle in the TRSB state. The transition form TRSB to the state with time-reversal symmetry takes place when the triangle relation is broken. From eqs.(94) and (95), the critical temperature T_c should satisfy

$$N_j \ln\left(\frac{2e^{\gamma_E}\omega_{Dj}}{\pi k_B T_c}\right) = \gamma_{jj} - \frac{\gamma_{jn}\gamma_{jm}}{\gamma_{nm}}, \tag{97}$$

where j, n and m are different to one another and γ_E is the Euler constant. The stability of TRSB state has been examined by evaluating the free energy[38,47,91]

In the simplest case where all the bands are equivalent and γ_{ij} $(i \neq j)$ are the same, the chiral state in Fig.3 is realized. We have $(\theta_1, \theta_2, \theta_3) = (0, 2\pi/3, 4\pi/3)$ for Fig.1(a) and $(\theta_1, \theta_2, \theta_3) = (0, 4\pi/3, 2\pi/3)$ for Fig.1(b). The two states are degenerate and have chirality $\kappa = 1$ and $\kappa = -1$, respectively, where the chirality is defined by $\kappa = (2/3\sqrt{3})[\sin(\theta_1 - \theta_2) + \sin(\theta_2 - \theta_3) + \sin(\theta_3 - \theta_1)]$. In the chiral state $\Delta_1/\gamma_{23}, \cdots$ form an equilateral triangle. In this case the eigenvalues of the gap equation are degenerate and the chiral TRSB state is realized.

For $N > 3$ it is not straightforward to derive the condition for TRSB. We consider here a separable form for the Josephson couplings:

$$\gamma_{ij} = \gamma_i\gamma_j \quad \text{for} \quad i \neq j, \tag{98}$$

where $\gamma_j (\neq 0)$ $(j = 1, \cdots, N)$ are real constants. The condition $\gamma_{12}\gamma_{23}\gamma_{31} = \gamma_1^2\gamma_2^3\gamma_3^2 > 0$ is satisfied. For $N = 4$ we obtain from eq.(90)

$$\frac{\Delta_1}{\gamma_2\gamma_3\gamma_4} + \frac{\Delta_2}{\gamma_3\gamma_4\gamma_1} + \frac{\Delta_3}{\gamma_4\gamma_1\gamma_2} + \frac{\Delta_4}{\gamma_1\gamma_2\gamma_3} = 0. \tag{99}$$

Then the triangle condition in eq.(96) is generalized to the polygon condition for general $N \geq 3$:

$$\frac{\Delta_1}{\gamma_2\gamma_3\cdots\gamma_N} + \frac{\Delta_2}{\gamma_3\gamma_4\cdots\gamma_N\gamma_1} + \cdots + \frac{\Delta_N}{\gamma_1\gamma_2\cdots\gamma_{N-1}} = 0. \tag{100}$$

We assume that the polygon is not crushed to a line, which means, in the case $N = 3$, the triangle inequality holds. Under these conditions, the solution with time-reversal symmetry breaking exists and massless excitation modes also exist at the same. The existence of massless modes will be examined in next section.

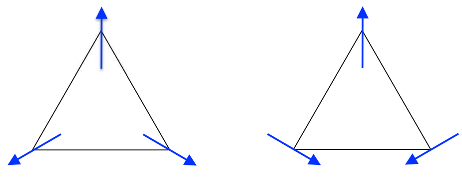

Fig. 3 Chiral state with time-reversal symmetry breaking. Two states have the chirality $\kappa = +1$ for (a) and $\kappa = -1$ for (b).

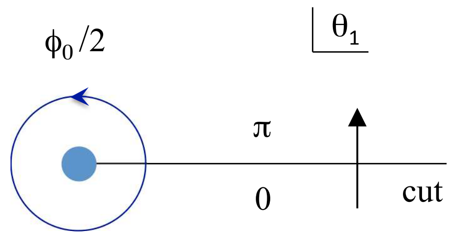

Fig. 4 Half-quantum flux vortex with a line singularity (kink). The phase variables θ_1 changes from 0 to π when crossing a singularity.

8 Half quantum-flux vortex and a Monopole

The sine-Gordon model has been studied to investigate a new dynamics of multi-gap superconductors[60, 61,112]. When the oscillation of phase difference φ is small, we can expand the potential around a minimum. This results in the Leggett mode as described in section III. In the presence of large oscillation, we cannot use a perturbative method and we must consider a non-perturbative kink solution. This leads to a half-quantum flux vortex.

The sine-Gordon model has a kink solution[81]. If we impose the boundary condition such that $\varphi \to 0$ as $x \to -\infty$ and $\varphi \to 2\pi$ as $x \to \infty$, we have a kink solution like $\varphi = \pi + 2\sin^{-1}(\tanh(\sqrt{\kappa}x))$ for a constant κ. The phase difference φ should be changed from 0 to 2π to across the kink. This means that θ_1 changes from 0 to π and at the same time θ_2 changes from 0 to $-\pi$. In this case, a half-quantum-flux vortex exists at the edge of the kink. This is shown in Fig.4 where the half-quantum vortex is at the edge of the cut (kink). A net change of θ_1 is 2π by a counterclockwise encirclement of the vortex, and that of θ_2 vanishes. Then, we have a half-quantum flux vortex.

The phase-difference gauge field $\mathbf{B}$ is defined as[51]

$$\mathbf{B} = -\frac{\hbar c}{2e^*}\nabla\varphi. \tag{101}$$

The half-quantum vortex can be interpreted as a monopole[42]. Let us assume that there is a cut, namely, kink on the real axis for $x > 0$. The phase θ_1 is represented by

$$\theta_1 = -\frac{1}{2}\mathrm{Im}\log\zeta + \pi, \tag{102}$$

where $\zeta = x + iy$. The singularity of θ_j can be transferred to a singularity of the gauge field by a gauge transformation. We consider the case $\theta_2 = -\theta_1$: $\varphi = 2\theta_1$. Then we have

$$\mathbf{B} = -\frac{\hbar c}{2e^*}\nabla\varphi = -\frac{\hbar c}{e^*}\frac{1}{2}\left(\frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2}, 0\right). \tag{103}$$

Thus, when the gauge field $\mathbf{B}$ has a monopole-type singularity, the vortex with half-quantum flux exists in two-gap superconductors.

Let us consider the fictitious z axis perpendicular to the x-y plane. The gauge potential (1-form) is given by

$$\Omega_\pm = -\frac{1}{2}\frac{1}{r(z\pm r)}(ydx - xdy) = \frac{1}{2}(\pm 1 - \cos\theta)d\phi, \tag{104}$$

where $r = \sqrt{x^2+y^2+z^2}$, and θ and ϕ are Euler angles. $\Omega_\pm$ correspond to the gauge potential in the upper and lower hemisphere $H_\pm$, respectively. $\Omega_\pm$ are connected by $\Omega_+ = \Omega_- + d\phi$. The components of Ω_+ are

$$\Omega_\mu = \frac{1}{2}(1-\cos\theta)\partial_\mu\phi. \tag{105}$$

At $z = 0$, Ω_μ coincides with the gauge field for half-quantum vortex. If we identify φ with ϕ, we obtain

$$\mathbf{B} = \frac{\hbar c}{e^*}\mathbf{\Omega}, \tag{106}$$

at $\theta = \pi/2$. $\{\Omega_\pm\}$ is the U(1) bundle P over the sphere S^2. The Chern class is defined as

$$c_1(P) = -\frac{1}{2\pi}F = -\frac{1}{2\pi}d\Omega_+. \tag{107}$$

The Chern number is given as

$$\begin{aligned} C_1 &= \int_{S^2} c_1 = -\frac{1}{2\pi}\int_{S^2} F \\ &= -\frac{1}{2\pi}\left(\int_{H_+} d\Omega_+ + \int_{H_-} d\Omega_-\right) = 1. \end{aligned} \tag{108}$$

In general, the gauge field $\mathbf{B}$ has the integer Chern number: $C_1 = n$. For n odd, we have a half-quantum flux vortex.

The half-flux vortex has been investigated in the study of p-wave superconductivity[59,113,114]. In the case of chiral p-wave superconductivity, the singularity of $U(1)$ phase is, however, canceled by the kink structure of the d-vector. This is the difference between two-band superconductivity and p-wave superconductivity.

As we can expect easily, a fractional quantum-flux vortex state is not stable because the singularity (kink, domain wall) costs energy being proportional to the square root of the Josephson coupling. Thermodynamic stability was discussed in Ref.[62]. Two vortices form a molecule by two kinks. This state may have lower energy than the vortex state with a single quantum flux ϕ_0 because the magnetic energy of two fractional vortices is smaller than ϕ_0^2 of the unit quantum flux. The energy of kinks is proportional to the distance R between two fractional vortices when R is large. Thus, the attractive interaction works between them when R is sufficiently large. There is an interesting analogy between quarks and fractional flux vortices[115].

9 Massless Nambu-Goldstone modes

We examined the phase modes that are Nambu-Goldstone modes by nature emerging due to a spontaneous symmetry breaking in section III. There, one mode becomes massive by coupling to the scalar potential, called the plasma mode, and the other modes become massive due to Josephson couplings, called the Leggett modes. In this section we show that massive modes change into massless modes when some conditions are satisfied.

The Josephson potential is given as

$$V \equiv \sum_{i\neq j}\gamma_{ij}\bar{\Delta}_i\bar{\Delta}_j\cos(\theta_i-\theta_j), \tag{109}$$

where $\gamma_{ij} = \gamma_{ji}$ are chosen real. Obviously the phase difference modes $\theta_i - \theta_j$ acquire masses. This would change qualitatively when N is greater than 3 or equal to 3. We discuss this in this section.

We show that massless modes exist for an N-equivalent frustrated band superconductor. Let us consider the potential for $N \geq 4$ given by

$$\begin{aligned} V = \Gamma[&\cos(\theta_1-\theta_2) + \cos(\theta_1-\theta_3) + \cdots + \cos(\theta_1-\theta_N) \\ &+ \cdots + \cos(\theta_{N-1}-\theta_N)]. \end{aligned} \tag{110}$$

For $\Gamma > 0$, there are two massive modes and $N-3$ massless modes, near the minimum $(\theta_1, \theta_2, \theta_3, \theta_4, \cdots) = (0, 2\pi/N, 4\pi/N, 6\pi/N, \cdots)$. This can be seen by writing the potential in the form

$$V = \frac{\Gamma}{2}\Big[\Big(\sum_{i=1}^N \mathbf{S}_i\Big)^2 - N\Big], \tag{111}$$

where $\mathbf{S}_i$ $(i = 1, \cdots, N)$ are two-component vectors with unit length $|\mathbf{S}_i| = 1$. V has a minimum $V_{min} = -\Gamma N/2$ for $\sum_i \mathbf{S}_i = 0$. Configurations under this condition have the same energy and can be continuously mapped to each other with no excess energy. At $(\theta_1, \theta_2, \cdots) = (0, 2\pi/N, 4\pi/N, \cdots)$ with $V = -\Gamma N/2$, satisfying $\sum_i \mathbf{S}_i = 0$, the vectors $\mathbf{S}_i$ form a polygon. The polygon can be deformed with the same energy (see Figs.5(a) and 5(b)). The existence of massless modes was examined numerically for the multi-gap BCS model[53]. It has been

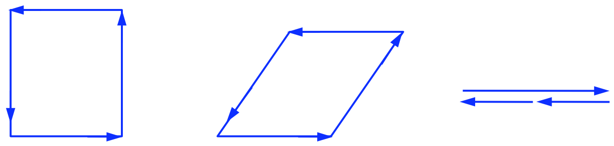

Fig. 5 Polygon state satisfying $\sum_j \mathbf{S}_j = 0$ for $N = 4$ in (a) and (b), where time-reversal symmetry is broken and a massless mode exists. A linear state with $\sum_j \gamma_j \Delta_j = 0$ is shown in (c), where a massless mode exists but the time-reversal symmetry is not broken.

shown that there is a large region in the parameter space where massless modes exist.

Let us discuss the Josephson potential in a separable form. This is given by

$$V = \sum_{i\neq j} \gamma_{ij} \Delta_i^* \Delta_j = \sum_{i\neq j} \gamma_i \gamma_j \Delta_i^* \Delta_j. \quad (112)$$

This is written as

$$V = |P|^2 - \sum_j \gamma_j^2 |\Delta_j|^2, \quad (113)$$

where $P = \sum_j \gamma_j \Delta_j$. V has a minimum when $P = 0$ is satisfied. $P = 0$ is equivalent to the polygon condition in eq.(100). Because the polygon for $N > 3$ can be deformed continuously without finite excitation energy, a massless mode exists[51] (Figs.5 (a) and (b)). We have one massless mode for $N = 4$ and two massless modes for $N = 5$. A spin model, corresponding to the Josephson model considered here, also has gapless excitation modes.

When the polygon is crushed to a line, the time-reversal symmetry is not broken. A massless mode, however, exists when $P = 0$. An example is shown in Fig.5(c) called a linear model. In this model there are two independent modes and the quadratic term of one mode vanishes as can be shown by explicit calculations. A mode called the scissor mode becomes massless.

Although we did not consider an effect of the amplitude mode (Higgs mode) η_j, this mode may be important when discussing the stability of massless modes. This is a future problem.

10 Sine-Gordon model

10.1 (d+1)D sine-Gordon model

The phase difference mode is described by the sine-Gordon model. For the two-band model with equivalent bands, the Lagrangian density is

$$\mathcal{L}_E = \frac{1}{2}\rho_F(\partial_\tau\varphi)^2 + \frac{n}{4m}(\nabla\varphi)^2 + 2\gamma_{12}\bar{\Delta}_1\bar{\Delta}_2\cos(2\varphi), \quad (114)$$

where φ is the half of the phase difference of the gap functions. Since $\rho_F \sim k_F^{d-2}$ and $n \sim k_F^d$ for the Fermi wave number k_F, where d is the space dimension, we write the action of this model in the form:

$$S_{SG} = \frac{\Lambda^{d-1}}{g}\int_0^\beta dx_0 \int d^d x \Big[\frac{1}{2}(\partial_\mu\varphi)^2 - \alpha\Lambda^2\cos\varphi\Big], \quad (115)$$

where we redefined 2φ to φ. We set $x_0 = Av_F\tau$ for a constant A and $\beta = Av_F/(k_BT)$. Λ is a cutoff and g and α are coupling constants. α is proportional to the strength of the Josephson coupling $|\gamma_{12}|$. We adopt that α is positive; otherwise we consider $|\alpha|$. WE define the dimensionless inverse temperature u by $\beta = u/\Lambda$. The action S_{SG} has a factor Λ^{d-1}, so that the coupling constant g is dimensionless. Thus, the phase difference mode is modeled by the (d+1)D sine-Gordon model. In the limit of small β (high-temperature limit), the model is reduced to the d-dimensional sine-Gordon model:

$$S_{SG}^d = \frac{\Lambda^{d-2}}{t}\int d^d x\Big[\frac{1}{2}(\partial_\mu\varphi)^2 - \alpha\Lambda^2\cos\varphi\Big], \quad (116)$$

where we set $t \equiv g/u$.

10.2 Renormalization group equation

Let us investigate the renormalization group flow of the (d+1)D sine-Gordon model on the basis of the Wilson renormalization group method[116,117] at finite temperature[118,119]. In general, the Josephson coupling is small and thus the results will be relevant in the region where the Josephson coupling is still small and finite. We neglect the effect of the renormalization on T_c.

The renormalization group equations are

$$\Lambda\frac{\partial g}{\partial\Lambda} = (d-1)g + c(u)\alpha^2 g\coth\left(\frac{u}{2}\right), \quad (117)$$

$$\Lambda\frac{\partial}{\partial\Lambda}\frac{\alpha}{g} = -\left(d+1-g\frac{\Omega_d}{4(2\pi)^d}\coth\left(\frac{u}{2}\right)\right)\frac{\alpha}{g}, \quad (118)$$

$$\Lambda\frac{\partial u}{\partial\Lambda} = u. \quad (119)$$

Here, $c(u)$ is a constant for large u and is proportional to u for small u; $c(u) = c_1 u$. Ω_d is the solid angle in d dimensions.

Let us consider fluctuations of the Leggett mode φ. A typical fluctuation mode is the kink (soliton) excitation where φ changes from 0 to 2π or 2π to 0 in some regions in a superconductor. The one-dimensional kink, namely the domain wall, is expected to appear easily due to quantum fluctuation. The one dimensional means that $\varphi(\tau, x_1, \cdots)$ depends on only one space variable x_1. The renormalization group equations for $d = 1$

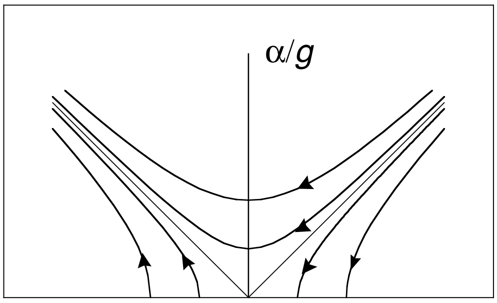

Fig. 6 Renormalization group flow in the plane of g and α/g (d=1). Arrows indicate the direction of flow when the cutoff decreases.

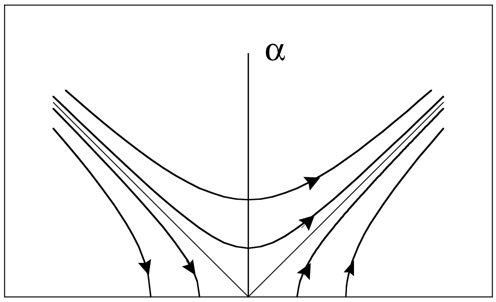

Fig. 7 Renormalization group flow for the $N = 3$ generalized sine-Gordon model ($d = 2$). The flow is indicated as μ increases ($\mu \to \infty$).

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$$\Lambda \frac{\partial g}{\partial \Lambda} = c(u)\alpha^2 g \coth\left(\frac{u}{2}\right), \tag{120}$$

$$\Lambda \frac{\partial}{\partial \Lambda}\frac{\alpha}{g} = -\left(2 - \frac{g}{4\pi}\coth\left(\frac{u}{2}\right)\right)\frac{\alpha}{g}. \tag{121}$$

At low temperature where $u = \beta\Lambda \gg 1$, there is a fixed point at $g = 8\pi$ and $\alpha = 0$. We show the renormalization group flow when the cutoff Λ decreases in Fig.6. As the cutoff Λ is reduced, $g(> 0)$ also decreases.

At high temperature, the equations reduce to

$$\Lambda \frac{\partial \Lambda}{\partial \Lambda} = (d-2)t + 2c_1\alpha^2 t, \tag{122}$$

$$\Lambda \frac{\partial \alpha}{\partial \Lambda} = -\alpha\left(2 - \frac{\Omega_d}{2(2\pi)^d}t\right). \tag{123}$$

There is a fixed point at $t = 8\pi$ and $\alpha = 0$ for $d = 2$. If this set of equations can be applied to a two-gap superconductor, there is a Kosterlitz-Thouless-like transition at $t = k_B T g/\Lambda = 8\pi$.

11 Chiral transition

11.1 N-variable sine-Gordon model

In this section we present a field theoretic model that shows a chiral transition. This model is extracted from a model for multi-gap superconductors. The model should be regarded as a model in field theory, and we also discuss applicability to real superconductors. We adopted the London approximation to derive the model, where the fluctuation modes (Higgs modes) η_j of the gap functions are neglected. A role of the fluctuation mode concerning the existence of the phase transition would be a problem for future discussion.

Let us consider an action for phase variable θ_j:

$$S[\theta] = \frac{1}{k_B T}\int d^d x\Big[\sum_j \frac{n_{sj}}{2m_j}(\nabla\theta_j)^2 + \sum_{i\neq j}\gamma_{ij}\bar{\Delta}_i\bar{\Delta}_j\cos(\theta_i - \theta_j)\Big], \tag{124}$$

where we neglect τ dependence of θ_j. We simply assume that $K_j \equiv n_{sj}/(2m_j) = K$, $\bar{\Delta}_j = \bar{\Delta}$ and $\gamma_{ij} = \gamma_{ji} = \gamma$, namely, all the bands are equivalent. Then the action for the phase variables θ_j is

$$S[\theta] = \frac{\Lambda^{d-2}}{t}\int d^d x\left(\sum_j(\nabla\theta_j)^2 + \alpha\Lambda^2\sum_{i<j}\cos(\theta_i-\theta_j)\right), \tag{125}$$

where $t/\Lambda^{d-2} = k_B T/K$ and $\lambda\Lambda^2 = 2\gamma\Delta_0^2/K$. We have introduced the cutoff Λ so that t and α are dimensionless parameters. We assume that $\alpha > 0$ in this paper. We consider the case $N = 3$ and discuss the phase transition in this model. Apparently this model has S_3 symmetry. If we neglect the kinetic term, the ground states is two-fold degenerate. The two ground states are indexed by the chirality κ.

We perform a unitary transformation: $\theta_1 = -2\pi/3 - (1/\sqrt{2})\eta_1 + (1/\sqrt{6})\eta_2 + (1/\sqrt{3})\eta_3$, $\theta_2 = -(2/\sqrt{6})\eta_2 + (1/\sqrt{3})\eta_3$, and $\theta_3 = 2\pi/3 + (1/\sqrt{2})\eta_1 + (1/\sqrt{6})\eta_2 + (2/\sqrt{3})\eta_3$, where η_i ($i = 1,2,3$) indicate fluctuation fields. η_3 describes the total phase mode, $\eta_3 = (\theta_1 + \theta_2 + \theta_3)/\sqrt{3}$, and is not important because this mode turns out to be a plasma mode by coupling with the long-range Coulomb potential. The action $S[\eta] \equiv S[\theta]$ becomes

$$S[\eta] = \frac{\Lambda^{d-2}}{t}\int d^d x\Big[\sum_j(\nabla\eta_j)^2 + \alpha\Lambda^2\Big(\cos\left(\sqrt{2}\eta_1 + \frac{4\pi}{3}\right) + 2\cos\left(\frac{1}{\sqrt{2}}\eta_1 + \frac{2\pi}{3}\right)\cos\left(\sqrt{\frac{3}{2}}\eta_2\right)\Big)\Big]. \tag{126}$$

This model shows the chiral transition[55] as well as the Kosterlitz-Thouless transition[120]. The renormalization group method[121,122] is applied to obtain the beta functions. They are given by

$$\mu\frac{\partial t}{\partial\mu} = (d-2)t + At\alpha^2 \tag{127}$$

$$\mu\frac{\partial\alpha}{\partial\mu} = -2\alpha + \frac{1}{4\pi}\alpha t, \tag{128}$$

for the mass parameter μ. Here A is a constant. The equation for α has a fixed point at $t = 8\pi$. In two dimension $d = 2$ the renormalization group flow is the same as that for the Kosterlitz-Thouless transition (see Fig.7).

11.2 Chirality Transition

There is a chirality transition at a finite temperature where the states with chirality $\kappa = \pm 1$ disappear and simultaneously the chirality vanishes. This is shown by taking account of the fluctuation around the minimum of the potential. Using $\cos(\sqrt{3/2}\eta_2) = 1-(4/3)\eta_2^2+\cdots$, the action is written as

$$\begin{aligned} S = \frac{\Lambda^{d-2}}{t}\int d^dx\Big[\sum_j(\nabla\eta_j)^2 + \alpha\Lambda^2\Big(\cos\left(\sqrt{2}\eta_1+\frac{4\pi}{3}\right) \\ -2\Big|\cos\left(\frac{1}{\sqrt{2}}\eta_1+\frac{2\pi}{3}\right)\Big|\Big) \\ +\frac{3\alpha\Lambda^2}{2}\Big|\cos\left(\frac{1}{\sqrt{2}}\eta_1+\frac{2\pi}{3}\right)\Big|\eta_2^2\Big]. \end{aligned} \tag{129}$$

We integrate out the field η_2 to obtain the effective action. The effective free-energy density in two dimensions is obtained as

$$\begin{aligned} \frac{f[\varphi]}{\Lambda^2} &= \frac{1}{2}K\Lambda^{-2}(\nabla\varphi)^2 + \epsilon_0\Big(\cos\varphi - 2\Big|\cos\left(\frac{\varphi}{2}\right)\Big|\Big) \\ &+ \frac{1}{2}k_BT\frac{c}{4\pi}\ln\left(\frac{c\Lambda^d}{t}+\frac{3\alpha\Lambda^d}{2t}\Big|\cos\left(\frac{\varphi}{2}\right)\Big|\right) \\ &+ k_BT\frac{3\alpha}{16\pi}\Big|\cos\left(\frac{\varphi}{2}\right)\Big|\ln\Big(1+\frac{2c}{3\alpha}\Big|\cos\left(\frac{\varphi}{2}\right)\Big|^{-1}\Big), \end{aligned} \tag{130}$$

for $\varphi \equiv 4\pi/3+\sqrt{2}\eta_1$ where Λ is a cutoff, c is a constant and $\epsilon_0 = k_BT\alpha/t = 2\gamma\Delta_0^2/\Lambda^2$. The critical temperature T_{chiral} of the chirality transition is determined by the condition that we have a minimum at $\varphi = \pi$ (first-order transition). T_{chiral} is shown as a function of α in Fig.8. $T_{chiral} = (K/k_B)t_c$ is dependent on α, where α is proportional to the Josephson coupling, while the temperature of the Kosterlitz-Thouless transition $T_{KT} = (K/k_B)8\pi$ is independent of α. Thus T_{chiral} and T_{KT} are different in general.

We have shown a model which shows a transition due to growing fluctuations. The disappearance of the chirality results in the emergency of a Nambu-Goldstone boson. This represents the phenomenon that the Nambu-Goldstone boson appears from a fluctuation effect. Please note that this does not say that a discreet symmetry can be broken by Nambu-Goldstone boson proliferation. A Nambu-Goldstone would emerge as a result of a discree symmetry breaking. At $T > T_{chiral}$ two spins in Fig.3 are antiferromagnetically aligned and one spin vanishes. This means that the one spin is rotating freely accompanied with the existence of a massless boson. Our model shows that the Z_2-symmetry breaking induces a massless boson. If we neglect the kinetic term in the action, T_{chiral} is determined uniquely as $T_{chiral} = \epsilon_0/2$. ϵ_0 corresponds to J in the two-dimensional XY model. This suggests that there is a chirality transition in the 2D XY model on a two-dimensional triangular lattice at near $T = J/2$, which has been confirmed by a numerical simulation[125]. The existence of the Kosterlitz-Thouless transition has also been shown at near $T = J/2$.

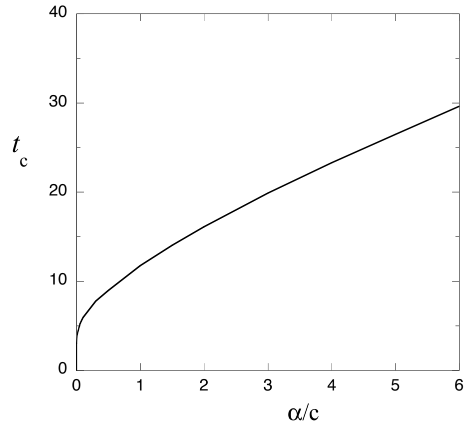

Fig. 8 $t_c \equiv k_BT_{chiral}/K$ as a function of α/c with $c = 4\pi$.

We discuss whether our model is applicable to real superconductors. We expect that our model applies to, for example, layered superconductors like cuprates with small Josephson couplings. This type of transition has been discussed for three-band superconductors with frustrated interband Josephson couplings[123]. Recent experiments have indicated that a first-order phase transition below the superconducting transition temperature occurs in multilayer cuprate superconductor $HgBa_2Ca_4Cu_5O_y$[124]. We hope that this phase transition is related to the dynamics of multicomponent order parameters.

12 $SU(N)$ sine-Gordon model

In this section let us consider a generalized Josephson interaction where the Josephson term is given by a G-valued sine-Gordon potential for a compact Lie group G. This model includes multiple excitation modes, and is a nonabelian generalization of the sine-Gordon model. The Lagrangian is written as

$$\mathcal{L} = \frac{1}{2t}\mathrm{Tr}\partial_\mu g \partial^\mu g^{-1} + \frac{\alpha}{2t}\mathrm{Tr}(g + g^{-1}), \tag{131}$$

for $g \in G$. When $g = e^{i\varphi} \in U(1)$, this Lagrangian is reduced to that of the conventional sine-Gordon model. This model can be regarded as the chiral model with the mass term. Here we consider the $SU(N)$ or $O(N)$ model: $G = SU(N)$ or $O(N)$. In the limit $t \to \infty$ with keeping $\lambda \equiv \alpha/t$ constant, the $SU(N)$ sine-Gordon model is reduced to a unitary matrix model. It has been shown by Gross and Witten that, in the large N limit with the coupling constant $\lambda = N\beta$, for the model $N\beta\mathrm{Tr}(g + g^\dagger)$, there is a third-order transition at some critical t_c[126]. Brezin and Gross considered the model to generalize the coupling constant λ to be a matrix and also found that there is a phase transition[127–129]. Recently, the vortex structure for a nonabelian sine-Gordon model was investigated numerically[130].

An element $g \in G$ is represented in the form:

$$g = g_0 \exp\left(i\lambda \sum_a T_a \pi_a\right), \tag{132}$$

where λ is a real number $\lambda \in \mathbf{R}$ and $g_0 \in G$ is a some element in G. We put $g_0 = 1$ in this paper. T_a $(a = 1, 2, \cdots, N_T)$ form a basis of the Lie algebra of G where $N_T = N^2 - 1$ for $SU(N)$ and $N_T = N(N-1)/2$ for $O(N)$. $\{T_a\}$ are normalized as

$$\mathrm{Tr}T_a T_b = c\delta_{ab}, \tag{133}$$

with a real constant c. The scalar fields π_a indicate fluctuations around the classical solution, that is, the nonabelian perturbation to the state g_0. We expand g by means of π_a as

$$g = g_0\Big[1 + i\lambda T_a\pi_a - \frac{1}{2}\lambda^2(T_a\pi_a)^2 + \cdots\Big], \tag{134}$$

and evaluate the beta functions of renormalization group theory.

The renormalization group equations read[92]

$$\mu\frac{\partial t}{\partial \mu} = (d-2)t - \frac{C_2(G)}{2}t^2 + A_0 C(N)t\alpha^2, \tag{135}$$

$$\mu\frac{\partial \alpha}{\partial \mu} = -\alpha\left(2 - C(N)t\right), \tag{136}$$

where $A_0 = A_0(N)$ is a constant (depending on N) and the volume element $\Omega_d/(2\pi)^d$ is included in the definition of t for simplicity. $C(N)$ is the Casimir invariant in the fundamental representation given by

$$C(N) = c\frac{N^2-1}{N} \quad \text{for} \quad G = SU(N), \tag{137}$$

$$= c\frac{N-1}{2} \quad \text{for} \quad G = 0(N). \tag{138}$$

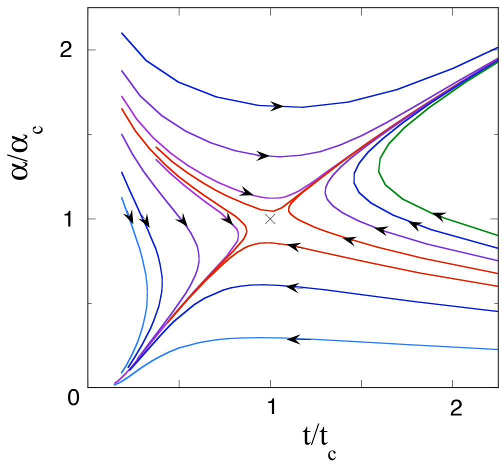

Fig. 9 Renormalization flow as μ increases. The cross indicates the bifurcation point.

The coefficient of t^2 term in $\mu\partial t/\partial\mu$ is the Casimir invariant in the adjoint representation defined by

$$\sum_{ab} f_{abc}f_{abd} = C_2(G)\delta_{cd}. \tag{139}$$

$C_2(G)$ is given as

$$C_2(G) = 2Nc \quad \text{for} \quad G = SU(N), \tag{140}$$

$$= (N-2)c \quad \text{for} \quad G = O(N), \tag{141}$$

$$\tag{142}$$

Thus beta functions are determined by Casimir invariants.

There is a zero of beta functions in two dimensions($d = 2$):

$$t_c = \frac{2}{C(N)}, \quad \alpha_c = \sqrt{\frac{C_2(G)}{A_0(N)}\frac{1}{C(N)}}. \tag{143}$$

This is a bifurcation point that divides the parameter space into two regions. One is the strong coupling region where $\alpha \to \infty$ as $\mu \to \infty$, and the other is the weak coupling region where $\alpha \to 0$ as $\mu \to \infty$. In the weak coupling region, we can use a perturbation theory by expanding g by means of the fluctuation fields π_a. This results in the existence of multiple frequency modes. We expect that these modes may be observed. There may be a possibility to classify excitation modes using a group theory. We show the renormalization flow in Fig.9.

13 Summary

The Nambu-Goldstone-Leggett modes and the Higgs mode are typical fluctuation modes in multi-gap superconductors. We expect that they play an important role. We derived the effective action and showed the dispersion relations of these modes. The mass of the Higgs mode is proportional to the gap amplitude. The Nambu sum rule does not hold in general in a multi-gap superconductor. One mode among the Nambu-Goldstone modes becomes the massive mode in the presence of the Coulomb potential (Higgs mechanism). An N-gap superconductor has $N-1$ phase-difference variables, and the $U(1)^{N-1}$ phase invariance can be partially or totally broken spontaneously. The $N-1$ phase modes become in general massive due to the symmetry breaking by the Josephson interaction. When the Josephson couplings are frustrated, symmetry is partially broken and some of phase modes can be massless modes. A kink solution exists in the phase space of gap functions.

The kink solution provides a new excitation mode. A fractionally quantized flux vortex can exist at the edge of the kink in a magnetic field. The half-flux vortex can be regarded as a monopole with the Chern number in a superconductor. We discussed several versions of the sine-Gordon model and derived the renormalization group equations for these models. An effect of fluctuation is investigated, on the basis of a toy model, where the fluctuation restores the time reversal symmetry in the ground state with time-reversal symmetry breaking.

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References

1. V. A. Moskalenko: Fiz. Metal and Metallored 8, 2518 (1959).
2. H. Suhl, B. T. Mattis and L. W. Walker: Phys. Rev. Lett. 3, 552 (1959).
3. J. Peretti, Phys. Lett. 2, 275 (1962).
4. J. Kondo: Prog. Theor. Phys. 29, 1 (1963).
5. J. Bardeen, L. Cooper and R. Schrieffer: Phys. Rev. 106, 162 (1957).
6. G. Binning, A. Baratoff, H. E. Hoenig and J. G. Bednorz: Phys. Rev. Lett. 45, 1352 (1980).
7. L. P. Gorkov: arXiv:1508.00529.
8. S. V. Shulga et al.: Phys. Rev. Lett. 80, 1730 (1998).
9. J. Nagamatsu, N. Nakagawa, T. Muranaka, Y. Zenitani and J. Akimitsu: Nature 410, 63 (2001).
10. Y. Kamihara, T. Watanabe, M. Hirano and H. Hosono: J. Ame. Chem. Soc. 130, 3296 (2008).
11. *Superconductivity* Vols. I and II, ed. K. H. Bennemann and J. B. Ketterson (Springer, Berlin, 2008).
12. A. Bianconi, Nature Phys, 9, 536 (2013).
13. A. Bianconi, A. C. Castellano, M. De. Santis, P. Delogu, A. Garano and R. Giorgi, Solid State Commun. 63, 1135 (1987).
14. *The Physics of superconductors* Vols. I and II, ed. K. H. Bennemann and J. B. Ketterson (Springer, Berlin, 2003).
15. V. J. Emery, Phys. Rev. Lett. 58, 2794 (1987).
16. J. E. Hirsch, D. Loh, D. J. Scalapino and S. Tang, Phys. Rev. B39, 243 (1989).
17. K. Yamaji et al., Physica C304, 225 (1998).
18. T. Yanagisawa, S. Koike and K. Yamaji, Phys. Rev. B64, 184509 (2001).
19. T. Yanagisawa, S. Koike and K. Yamaji, Phys. Rev. B67, 132408 (2003).
20. C. Weber, A. Lauchi and G. A. Sawatzky, Phys. Rev. Lett. 102, 017005 (2009).
21. T. Yanagisawa, J. Phys. Soc. Jpn. 85, 114707 (2016).
22. S. Koike et al., J. Phys. Soc. Jpn. 68, 1657 (1999).
23. K. Yamaji et al., Physica C235, 2221 (1994).
24. T. Yanagisawa, Y. Shimoi and K. Yamaji, Phys. Rev. B52, R3860 (1995).
25. A. Bussmann-Holder, H. Keller, R. Khasanov, A. Simon, A. Bianconi and A. R. Bishop: New J. Phys. 13, 093009 (2011).
26. H. Y. Choi et al., Phys. Rev. B80, 052505 (2009).
27. P.M. Shirage, K. Kihou, K. Miyazawa, C.-H. Lee, H. Kito, H. Eisaki, T. Yanagisawa, Y. Tanaka. A. Iyo: Phys. Rev. Lett. 103, 257003 (2009).
28. T. Yanagisawa, K. Odagiri, I. Hase, K. Yamaji, P. M. Shirage, Y. Tanaka, A. Iyo and H. Eisaki, J. Phys. Soc. Jpn. 78, 094718 (2009).
29. A. D. Hillier, J. Quintanilla and R. Cywinskii, Phys. Rev. Lett. 102, 117007 (2009).
30. I. Hase and T. Yanagisawa, J. Phys. Soc. Jpn. 78, 084724 (2009).
31. A. D. Hillier, J. Quintanilla, B. Mazidian, J. F. Annett and R. Cywinskii, Phys. Rev. Lett. 109, 097001 (2012).
32. I. Hase and T. Yanagisawa, J. Phys. Soc. Jpn. 81, 103704 (2012).
33. D. J. Singh, Phys. Rev. B86, 174507 (2012).
34. V. K. Pecharsky, L. L. Miller and K. A. Gschneider, Phys. Rev. B58, 497 (1998).
35. Y. Iwamoto, Y. Iwasaki, K. Ueda and T. Kohara, Phys. Lett. A250, 439 (1998).
36. H. H. Sung, S. Y. Chou, K. J. Syu and W. H. Lee, J. Phys. Condens. Matter 20, 165207 (2008).
37. S. Katano et al., Phys. Rev. B90, 220508 (2014).
38. V. Stanev and Z. Tesanovic, Phys. Rev. B81, 134522 (2010).
39. Y. Tanaka and T. Yanagisawa, J. Phys. Soc. Jpn. 79, 114706 (2010).
40. Y. Tanaka and T. Yanagisawa, Solid State Commun. 150,1980 (2010).
41. R. G. Dias and A. M. Marques, Supercond. Sci. Technol. 24, 085009 (2011).
42. T. Yanagisawa, Y. Tanaka, I. Hase and K. Yamaji: J. Phys. Soc. Jpn. 81, 024712 (2012).
43. X. Hu and Z. Wang, Phys. Rev. B85, 064516 (2012).
44. V. Stanev, Phys. Rev. B85, 174520 (2012).
45. C. Platt, R. Thomale, C. Homerkamp and S. C. Zhang, Phys. Rev. B85, 180502 (2012).
46. S. Maiti and A. V. Chubukov, Phys. Rev. B87, 144511 (2013).
47. B. J. Wilson and M. P. Das, J. Phys. Condens. Matter 25, 425702 (2013).
48. Y. Takahashi, Z. Huang and X. Hu, J. Phys. Soc. Jpn. 83, 034701 (2014).
49. R. Ganesh, G. Baskaran, J. van den Brink and D. V. Efremov, Phys. Rev. Lett. 113, 177001 (2014).
50. Y. S. Yerin, A. N. Omelyanchouk and E. ll'chev, arXiv: 1503.04653 (2015).
51. T. Yanagisawa and I. Hase, J. Phys. Soc. Jpn. 82, 124704 (2013).
52. S. Z. Lin and X. Hu, New J. Phys. 14, 063021 (2012).

53. K. Kobayashi, M. Machida, Y. Ota and F. Nori, Phys. Rev. B88, 224516 (2013).
54. T. Koyama, J. Phys. Soc. Jpn. 83, 074715 (2014).
55. T. Yanagisawa and Y. Tanaka, New J. Phys. 16, 123014 (2014).
56. Y. Tanaka et al., Physica C516, 10 (2015).
57. S.-Z. Lin and X. Hu, Phys. Rev. Lett. 108, 177005 (2012).
58. Yu. A. Izyumov and V. M. Laptev: Phase Transitions 20, 95 (1990).
59. G. E. Volovik, *The Universe in a Helium Droplet* (Oxford University Press, Oxford, 2009).
60. Y. Tanaka: Phys. Rev. Lett. 88, 017002 (2002).
61. E. Babaev, Phys. Rev. Lett. 89, 067001 (2002).
62. S. V. Kuplevakhsky, A. N. Omelyanchouk and Y. S. Yerin: J. Low Temp. Phys. 37, 667 (2011).
63. A. J. Leggett: Prog. Theor. Phys. 36, 901 (1966).
64. Y. Tanaka et al.: Physica C471, 747 (2011).
65. V. Moshchalkov et al.: Phys. Rev. Lett. 102, 117001 (2009).
66. M. Silaev and E. Babaev, Phys. Rev. B84, 094515 (2011).
67. Y. Nambu, Phys. Rev. 117, 648 (1960).
68. Y. Nambu: in *Physics in the 21st Century* edited by K. Kikkawa, H. Kunitomo and H. Otsubo, World Scientific, Singapore (1997).
69. T. Yanagisawa, Novel Supercon. Materials 2015, 1 (2015).
70. B. T. Geilikman, R. O. Zaitsev and V. Z. Kresin, Sov. Phys.: Solid State 9, 642 (1967).
71. D. R. Tilley, Proc. Phys. Soc. 84, 573 (1964).
72. I. P. Ivanov. Phys. Rev. E79, 021116 (2009).
73. N. V. Orlova, A. A. Shanenko, M. V. Milosevic, F. M. Peeters, A. V. Vagov ad V. M. Axt, Phys. Rev. B87, 134510 (2013).
74. Z. Huang and X. Hu, Appl. Phys. Lett. 104, 162602 (2014).
75. Y. S. Yerin and A. N. Omelyanchouk, Low Temp. Phys. 40, 943 (2014).
76. Y. Tanaka, Super. Sci. Tech. 28, 034002 (2015).
77. J. G. Storey, J. W. Loram, J. R. Cooper, Z. Bukowski and J. Karpinski, Phys. Rev. B88, 144502 (2013).
78. R. G. Cai, L. Li, L. F. Li and Y. Q. Wang, JHEP 9, 074 (2013).
79. W.-Y. Wen, M.-S. Wu and S.-Y. Wu, Phys. Rev. D89, 066005 (2014).
80. M. Nishida, JHEP 08, 136 (2015).
81. R. Rajaraman, *Solitons and Instantons* (North-Holland, Elsevier, Amsterdam, 1987).
82. Y. Tanaka, J. Phys. Soc. Jpn. 70, 2844 (2001).
83. E. Babaev, L. D. Faddeev and A. J. Niemi, Phys. Rev. B65, 100512 (2002).
84. Y. M. Cho and P. Zhang, Phys. Rev. B73, 180506 (2006).
85. M. Eto, Y. Hirono, M. Nitta and S. Yasui, Prog. Theor. Exp. Phys. 2014, 012D01 (2014).
86. M. C. N. Fiolhais and J. L. Birman, Phys. Lett. A378, 2632 (2014).
87. E. Babaev and M. Speight, Phys. Rev. B72, 180502 (2005).
88. V. G. Kogan and J. Schmalian, Phys. Rev. B83, 054515 (2011).
89. E. Babaev and M. Silaev, Phys. Rev. B86, 016501 (2012).
90. V. G. Kogan and J. Schmalian, Phys. Rev. B86, 016502 (2012).
91. M. Marciani, L. Fanfarillo, C. Castellani and L. Benfatto: Phys. Rev. B88, 214508 (2013).
92. T. Yanagisawa, EPL 113, 41001 (2016).
93. R. Kleiner, F. Steinmeyer, G. Kunkel and P. Muller, Phys. Rev. Lett. 68, 2349 (1992).
94. G. Ohya, N. Aoyama, A. Irie, S. Kishida and H. Tokutaka, Jpn. J. Appl. Phys. 31, L829 (1992).
95. K. Tamasaku, Y. Nakamura and S. Uchida, Phys. Rev. Lett. 69, 1455 (1992).
96. Y. Matsuda, M. B. Gaifullin, K. Kumagai, K. Kadowaki and T. Mochiku, Phys. Rev. Lett. 75, 4512 (1995).
97. T. Koyama and M. Tachiki, Phys. Rev. B54, 16183 (1996).
98. S. G. Sharapov, V. P. Gusynin and H. Beck, Eur. Phys. J. B39, 062001 (2002).
99. H. J. Choi, D. Roundy, H. Sun, M. L. Cohen and S. G. Louie, Nature 418, 758 (2002).
100. G. Blumberg et al., Phys. Rev. Lett. 99, 014507 (2007).
101. Y. Ota, M. Machida, T. Koyama and H. Aoki, Phys. Rev. B83, 060507 (2011).
102. R. Matsunaga, Y. I. Hamada, K. Makise, Y. Uzawa, H. Terai, Z. Wang and R. Shimano, Phys. Rev. Lett. 111, 057002 (2013).
103. M. A. Measson, Y. Gallais, M. Cazayous, B. Clair, P. Rodiere, L. Cario and A. Sacuto, Phys. Rev. B89, 060503 (2014).
104. Y. Barlas and C. M. Varma, Phys. Rev. B87, 054503 (2013).
105. T. Koyama, Phys. Rev. B90, 064514 (2014).
106. N. Tsuji and H. Aoki, Phys. Rev. B92, 064508 (2015).
107. Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961).
108. Y. Nambu and G. Jona-Lasinio, Phys. Rev. 124, 246 (1961).
109. Y. Nambu, Physica D15, 147 (1985).
110. T. Koyama, J. Phys. Soc. Jpn. 88, 064715 (2016).
111. U. Welp et al., Phys. Rev. B78, 140510 (R) (2008).
112. A. Gurevich and V. Vinokur, Phys. Rev. Lett. 90, 047004 (2003).
113. H.-Y. Kee, Y. B. Kim and K. Maki, Phys. Rev. B62, R9275 (2000).
114. J. Jang et al., Science 331, 186 (2011).
115. M. Nitta, M. Eto, T. Fujimori and T. Ohashi, J. Phys. Soc. Jpn. 81, 084711 (2012).
116. K. G. Wilson and J. B. Kogut, Phys. Rep. 12C, 75 (1974).
117. J. B. Kogut, Rev. Mod. Phys. 51, 659 (1979).
118. C. Chakravarty, B. I. Halperin and D. R. Nelson, Phys. Rev. B39, 2344 (1989).
119. T. Yanagisawa, Phys. Rev. B46, 13896 (1992).
120. J. M. Kosterlitz and D. Thouless, J. Phys. C6, 1181 (1973).
121. J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, Oxford, 1989).
122. D. J. Amit, Y. Y. Goldschmidt and S. Grinstein, J. Phys. A: Math. Gen. 13, 585 (1980).
123. T. A. Bojesen, E. Babaev and A. Sudbo, Phys. Rev. B88, 220511 (2013).
124. Y. Tanaka et al., J. Phys. Soc. Jpn. 83, 074705 (2014).
125. S. Miyashita and H. Shiba, J. Phys. Soc. Jpn. 53, 1145 (1984).
126. D. J. Gross and E. Witten, Phys. Rev. D21, 446 (1980).
127. E. Brezin and D. J. Gross, Phys. Lett. B97, 120 (1980).
128. R. C. Brower and M. Nauenberg, Nucl. Phys. B180, 221 (1981).
129. E. Brezin and S. Hikami, JHEP 7, 67 (2010).
130. M. Nitta, Nucl. Phys. B895, 288 (1995).