A Class of Multi-Symmetric Polyhedra,

by

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In a paper[1] by the author it was found desirable to consider a class of polyhedra, called medial polyhedra, which possess only trihedral vertices, and only pentagonal and hexagonal faces. It is shown in that paper that if the faces are more than eleven in number, then there are exactly twelve pentagons and the remaining faces are hexagons.

Beginning with the regular dodecahedron one can obtain new medial polyhedra by regularly truncating vertices and chamfering edges, each new polyhedron possessing twelve regular pentagons whose centers coincide with the centers of the pentagons forming the original dodecahedron, while all the added faces are hexagons. It is proposed, in this paper, to determine all such medial polyhedra, composed of twelve congruent "patches," pentagonal in shape, each patch consisting of a regular pentagon at the center surrounded by a regular distribution of hexagons.

Since three patches meet in a vertex of a patch, symmetry requires that the vertex of a patch must be either the vertex of a polygon face, or the center of a polygon. Also, the middle of the side of a patch will be the vertex of a polygon, the center of a polygon, or the middle of the edge of a polygon. Each pentagonal patch may be divided into ten equivalent (congruent or symmetric) triangular patches. It suffices, then, to consider the disposition of the hexagons in this triangular patch to determine the polyhedra sought.

Topologically, the arrangement of the hexagons in a triangular patch is the same as in a 30° sector of a regular honeycomb arrangement of hexagons. Then there are as many of the sought polyhedra as there are topologically different positions of the vertex of a pentagonal patch in this 30° sector, that is, different with respect to the central pentagon.

Using a, b as the inclined coordinates (60° between axes) of the

(1) The Isoperimetric Problem for Polyhedra, Tôhoku Mathematical Journal, 40 (1934), 226-236.

vertex of a patch, the square of the distance from the center of the patch to the vertex is equal to a^2+ab+b^2. The total number of hexagons in this patch is therefore $10(a^2+ab+b^2-1)/12$. Adding the twelve pentagonal faces, the total number of faces bounding the polygon is: $n=10(a^2+ab+b^2-1)+12=10(a^2+ab+b^2)+2$.

In Fig. 1 the circles indicate the possible positions of the vertex of a patch; the number in the circle indicates the number of faces in the polyhedron using that patch.

Note that duplications for the value of n occur in the array (Fig. 1). The first duplication appears for $n=492$ and it arises from the coordinates 7,0 or 5,3. This, however, is not a duplication in

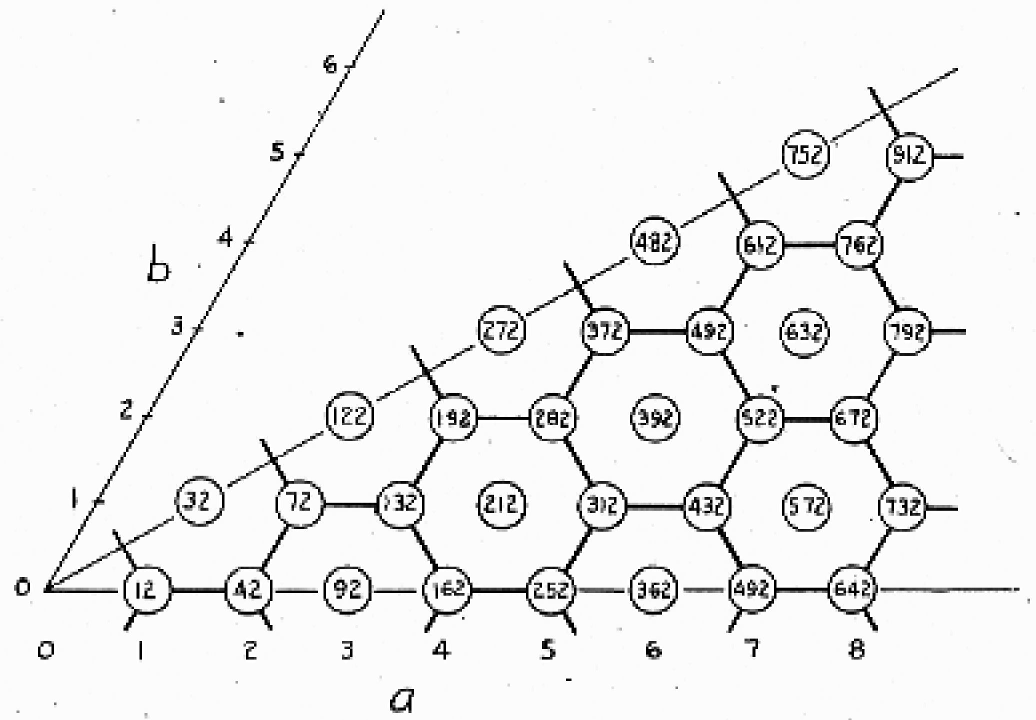


Fig. 1.

the polyhedra to which they refer since the polyhedra are topologically different. The difference can be seen readily in Fig. 2 and Fig. 3 by noting the chains (dotted lines) which proceed from the pentagons and join opposite edges of the hexagons. In one case the chains are terminated by pentagons while in the other they pass between the pentagons. Thus we come upon the remarkable fact that *trihedral polyhedra which possess the same number of hexagonal faces in addition to 12 regularly and symmetrically disposed pentagons can be topologically different.*

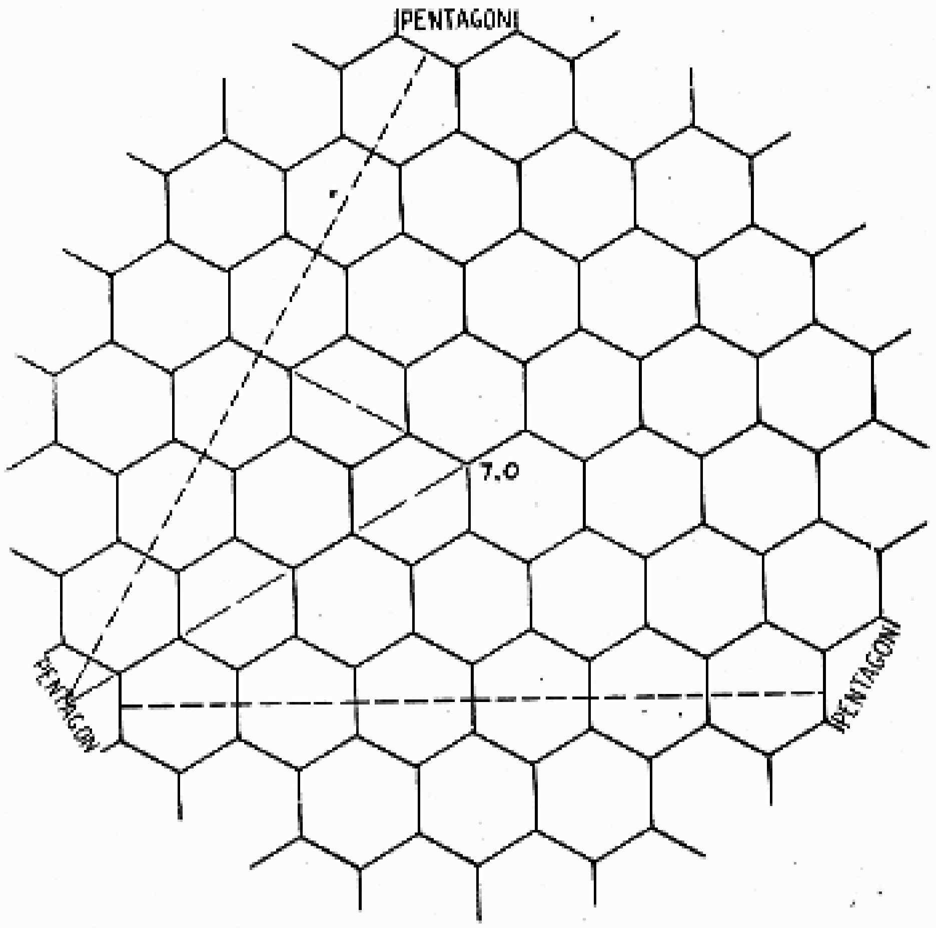
Repetitions in the array (Fig. 1) occur when there exists a multiplicity of pairs of positive integers a, b which generate the same R in (1):

(1) $$a^2+ab+b^2=R.$$

It is always possible to find an R for which there are at least as many solutions as any assigned number. We begin the demonstration of this fact by letting

(2) $$a=(3M+N)(M-N),$$

(3) $$b=4MN,$$



$a, b = 7,0$

Fig. 2.

where M and N are positive integers. Then,

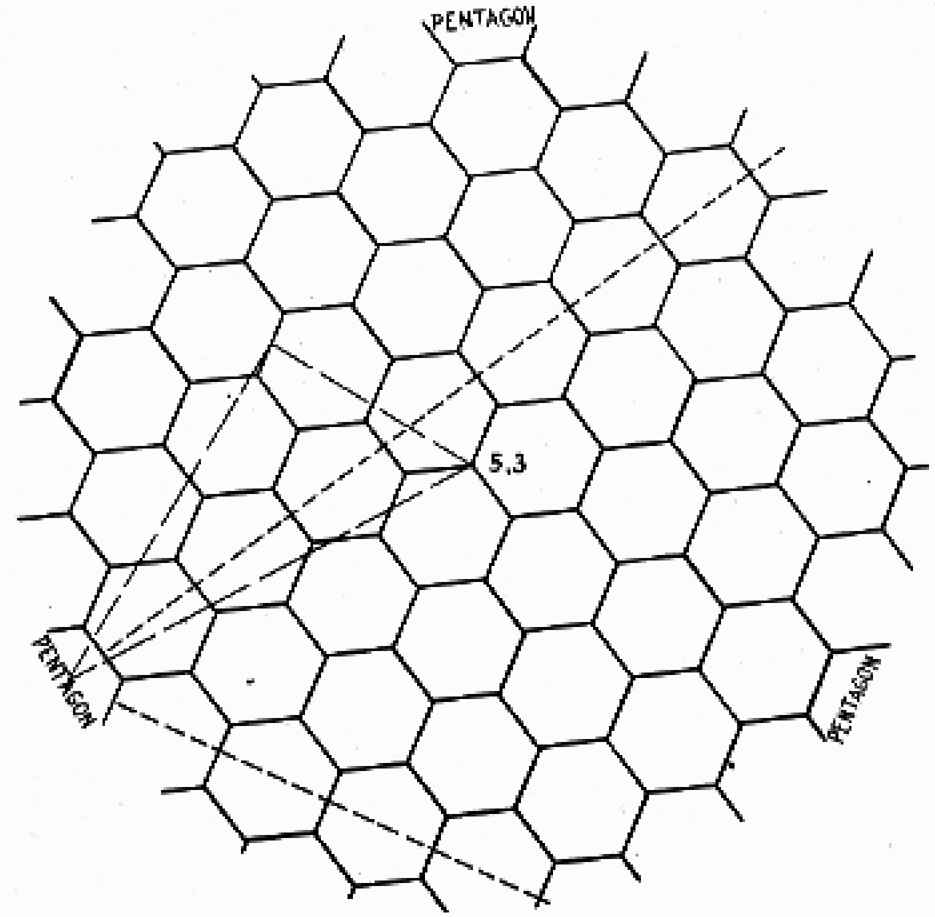
(4) $$R=(3M^2+N^2)^2\equiv c^2.$$

We are thus able to obtain an infinite set of values of R, in which each R is associated with at least two pairs of solutions of (1), namely a, b and c, 0. By combining two values of R we are able to obtain an R which has at least three pairs of solutions. For example, combining

(5) $$5^2+5\times3+3^2=7^2 \qquad (a, b=5, 3;\ c, 0=7,0)$$

with

(6) $$8^2+8\times7+7^2=13^2 \qquad (a, b=8, 7;\ c, 0=13,0)$$



$a, b = 5,3$

Fig. 3.

by multiplying (5) by 13^2 and (6) by 7^2

gives: $$65^2+65\times39+39^2=7^2\times13^2=91^2=8281$$

and $$56^2+56\times49+49^2=13^2\times7^2=91^2=8281.$$

It so happens that two other pairs appear as solutions so that for $R=8281$ we have the pairs: 91,0; 85,11; 80,19; 65,39; 56,49 which enable the construction of five topologically different polyhedra.

By combining values of R in the manner shown we are able to obtain an R which has a multiplicity of solutions exceeding in number any assigned number.

The phenomena of multiplicity described in the foregoing apply in the same way to other multi-symmetric polyhedra based on the tetrahedron and the cube. The pentagon at the origin in Fig. 1 and the pentagons in Fig. 2 and Fig. 3 are replaced by triangles to form the tetrahedral system, or by quadrilaterals to form the cubic system. The number of faces in the tetrahedral system is $2(a^2+ab+b^2)+2$, while in the cubic system it is $4(a^2+ab+b^2)+2$.

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On Certain Systems of Polynomials[1],

by

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1. *Introduction.* Nörlund[2] has introduced certain systems of polynomials, $B_\nu^{(n)}(x)$, $E_\nu^{(n)}(x)$, which generalize the classical Bernoulli and Euler polynomials. Just as these last are necessary for the study of difference equations of the first order, the generalized polynomials are needed in the theory of the equations of higher order. Recently, Milne-Thomson[3] has studied these in connection with generalizations of the Hermite polynomials. It is the purpose of this paper to study these systems further, using certain unifying concepts due to Appell[4], who has discussed a class of polynomials which includes, as special instances, all of the different types mentioned in the preceding. In the next section we summarize briefly the principal properties of polynomials of the Appell class, referring the reader for more details and comprehensive discussion to Appell's memoir.

2. *Appell Polynomials.* Let a sequence of constants $[\alpha_n] \equiv \alpha_0, \alpha_1, \ldots, \alpha_n, \ldots$ be given. The system of Appell polynomials $A_\nu(x)$ based on this sequence may be defined symbolically by

$$A_\nu(x) \equiv (x+\alpha)^\nu, \tag{1}$$

where α is the umbra of the given sequence, it being understood that after the binomial expansion has been performed, α^k is to be replaced by α_k. The characteristic property of the system is given by the mixed difference-differential equation

$$D_x A_\nu(x) = \nu A_{\nu-1}(x). \tag{2}$$

Usually the given sequence is defined by means of a generating function, so that

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(1) Presented in part to the American Mathematical Society, Nov. 30, 1934.

(2) Nörlund, Memoire sur les Polypomes de Bernoulli, Acta Mathematica, (1922), 121–196. See also his Differenzenrechnung, Chapter VI; (Springer, 1924).

(3) Milne-Thomson, Two Classes of Generalized Polynomials, Proc. London Math. Soc. 35 (1933).

(4) Appell, Sur une Classe de Polynomes, Ann. Sci. de L'Ecole Normale Super., (2) 9, (1880).

(See also Pincherle, Le Operazioni Distributive, p.p. 130–139, Bologna, 1901.) and Acta Mathematica, 10, 153 to 182. and 48, p. 279).