

The Isoperimetric Problem for Polyhedra,

by

Michael GOLDBERG, Washington, D. C., U. S. A.

1. *Introduction.* The problem of the determination of the convex polyhedra of greatest volume of a given number n of faces and a given surface area has been treated by Lhuillier⁽¹⁾, Steiner⁽²⁾, Lindelöf⁽³⁾, Kötter⁽⁴⁾, Minkowski⁽⁵⁾ and Steinitz⁽⁶⁾.

Lindelöf has shown that a necessary condition is that the faces of the polyhedron be tangent to a sphere at the centroid (center of gravity) of the faces. The problem may then be considered as the determination of the polyhedra of least volume circumscribed about a sphere. Steinitz has shown that the Lindelöf condition is sufficient for a polyhedron of a given type (that is, for a class of topologically isomorphic polyhedra) if all the vertices are trihedral. Minkowski proved the existence of the sought polyhedra.

In spite of the elementary nature of the problem, and the valient attempts at its solution by expert geometers, the definite results obtained are very meager. The complete census of the "best" polyhedra which these workers have determined is as follows:— (a) the regular tetrahedron, (b) the right triangular prism, (c) the cube, and (d) the right pentagonal prism. For $n > 7$, they left the problem practically untouched. Even for $n = 7$, the proof as given by Lindelöf is, in his own words, "only tentative".

⁽¹⁾ Lhuillier, S., De relatione mutua capacitatis et terminorum figurarum, etc., Varsaviae, (1782).

⁽²⁾ Steiner, J., Sur le maximum et le minimum des figures, J. Math., Berlin, vol. 24, (1842), pp. 93-152, 189-250 or Werke, Vol. 2, pp. 177-308.

⁽³⁾ Lindelöf, L., Propriétés générales des polyèdres etc., St. Petersburg, Bull. Ac. Sc. vol. XIV (1869), pp. 258-269. Recherches sur les polyèdres maxima, Helsingfors, Acta Soc. Sc. Fenn. vol. XXIV, No. 8 (1899).

⁽⁴⁾ Kötter, E., Volumen die kleinste Oberfläche besitzen, J. Math., Berlin, vol. 110 (1892), pp. 198-229.

⁽⁵⁾ Minkowski, H., Allgemeine Lehrsätze über die konvexen Polyeder, Göttingen, Nach. Ges. Wiss. Math.-phys. (1897), pp. 198-219, or Gesammelte Abhandlungen, vol. 2, pp. 103-121.

⁽⁶⁾ Steinitz, E., Polyeder und Raumeinteilungen, Encyk. der Math. Wiss., Berlin, Band III-12, Heft 9, (1922), pp. 38-43, 94-101. Über isoperimetrische Probleme bei konvexen Polyedern, J. Math., Berlin, vol. 158 (1927), pp. 129-153, vol. 159 (1928), pp. 133-143.

Beyond $n = 7$, the work of Steiner, Lindelöf, and Steinitz has been the consideration of pyramids, prisms, double pyramids, truncated pyramids, double skew pyramids, and truncated double skew pyramids. As n increases, we should expect that the "best" polyhedra would approach the sphere as a limit. But prisms, pyramids, and truncated pyramids, both single and double, do not even approach the sphere as a limit (the limit of double truncated pyramids is a double truncated cone); thus the number of "best" polyhedra that may be found among them must, of necessity, be few, and then only for small values of n . "Best" polyhedra are certainly not to be found among them for $n > 24$, and from evidence in this paper, very probably not for $n > 14$.

This paper presents a more general attack on the problem with a view to finding a clue where to seek the "best" polyhedra especially as n becomes large. The problem is beset with great difficulties because the number of possible polyhedra increases rapidly with n , and even the determination of the "best" of a single type of polyhedron can entail a formidable amount of work.

With Steinitz, let us define the number $K = F^3/V^2$ where F is the surface area and V is the volume of the polyhedron. Then the "best" polyhedra are those which have the smallest value of K . For polyhedra circumscribed about a sphere of unit radius, $K = 9F = 27V$. Steinitz develops formulae for K for pyramids, prisms, double pyramids, and double skew pyramids. The values of K thus obtained serve as upper bounds.

The only lower bound which Steinitz mentions is the value of K for the sphere. This paper develops closer lower bounds. For special cases, closer upper bounds are obtained thus narrowing the range in which K may be found.

2. *A Lower Bound,—The Circle Ideal.* Consider a polyhedron circumscribed about the unit sphere. Then consider a pyramid which has the center of the sphere as apex and one of the faces of the polyhedron as a base. This pyramid has unit altitude. Now construct a cone with the same solid angle at the apex. If the cone is made a right circular cone, then it is quite evident that the area of the base of the cone is made the least.

Now consider two pyramids the sum of whose apex angles is constant. Then the sum of the areas of the bases is the least when the pyramids become equal right circular cones. So then, if the

solid angle about a point, (4π) , is divided into n equal parts and each made the apex angle of a cone, a fictitious polyhedron of n faces may be made of these n cones, and the value of K ($K=9F$, that is, 9 times the sum of the areas of the bases) is less than for any real polyhedron. I call this value of K the *Circle Ideal* because the polygon faces become circles.

Let h be the height of the spherical segment which subtends a solid angle of $4\pi/n$ at the center of the sphere, and let θ be the half plane angle at the center of the sphere. Then, since the area of the convex part of the segment is $2\pi h$,

$$\frac{4\pi}{n} = 2\pi h \quad \text{or} \quad h = \frac{2}{n},$$

$$\cos \theta = 1 - h, \quad \tan \theta = \frac{\sqrt{2h - h^2}}{1 - h}.$$

Area of circle (base of cone) = $\pi \tan^2 \theta = \frac{\pi(2h - h^2)}{(1 - h)^2} = \frac{F}{n}$. Using $K(C)$ for the value of K obtained from the Circle Ideal

$$K(C) = \frac{F^3}{V^2} = 9F = \frac{9n\pi(2h - h^2)}{(1 - h)^2} = \frac{36n\pi(n - 1)}{(n - 2)^2}.$$

3. The Number of Edges per Face⁽¹⁾. Let m_1, m_2, \dots, m_n be the number of edges of the polygon faces of a convex polyhedron. Let M be their sum and m their mean value. Symbolically,

$$\sum_{i=1}^n m_i = M, \quad m = \frac{M}{n}.$$

Let e be the total number of edges, and v the total number of vertices, of the polyhedron. Then, since at least three edges meet in a vertex,

$$v \leq \frac{M}{3} \quad \text{and} \quad e = \frac{M}{2} = \frac{nm}{2}.$$

From Euler's law for polyhedra

$$n + v = e + 2,$$

$$n + \frac{M}{3} \geq \frac{M}{2} + 2,$$

$$n - 2 \geq \frac{M}{6} = \frac{nm}{6},$$

⁽¹⁾ See also Graustein, W. O., On the Average Number of Sides of Polygons of a Net, *Ann. Math.*, Princeton, N. J., 2nd series, vol. 32 (1931), pp. 149-153.

$$m \leq 6 - \frac{12}{n}.$$

4. The Pyramid Function. Let us find the area P of a regular m -gon tangent at its midpoint to a sphere of unit radius and which subtends the solid angle ϕ at the center of the sphere. Project the m -gon upon the sphere from the center of the sphere. Divide the plane and spherical m -gons each into $2m$ triangles by lines from the midpoint to the vertices and by perpendiculars from the midpoint to the sides. Then, in a spherical right triangle so formed, let A be the angle π/m , c be the hypotenuse, B be the other acute angle, and b be the side opposite to B . Then

Spherical excess of spherical right triangle

$$= A + B - \frac{\pi}{2} = \frac{\pi}{m} + B - \frac{\pi}{2} = \frac{\phi}{2m} = \text{area of spherical triangle.}$$

But $\cos b = \frac{\cos B}{\sin A}$, from which $\tan^2 b = \sin^2 A \sec^2 B - 1$,

Area of plane triangle = $\frac{1}{2} \tan A \tan^2 b = \frac{1}{2} \tan A (\sin^2 A \sec^2 B - 1) = \frac{P}{2m}$,

from which $P = m \tan A (\sin^2 A \sec^2 B - 1)$.

But $\phi = 2m(\pi/m + B - \pi/2) = 2\pi - 2m(\pi/2 - B)$,

or $B = \frac{\pi}{2} - \frac{\phi}{2m}$.

Let $G = \pi - \frac{\phi}{2}$, then $B = \frac{\pi}{2} - \frac{G}{m}$, $\sec^2 B = \csc^2 \frac{G}{m}$.

We can now write P as a function of m and ϕ

$$P(m, \phi) = m \tan \frac{\pi}{m} \left(\sin^2 \frac{\pi}{m} \csc^2 \frac{G}{m} - 1 \right), \quad \text{where } G = \pi - \frac{\phi}{2}.$$

Although this function was derived for integral values of m , henceforth we will consider $P(m, \phi)$ as a continuous function of m and ϕ .

It can be shown that the surface defined by $P(m, \phi)$ is everywhere concave upward. From this fact follows the

Lemma. If $P_1, m_1,$ and ϕ_1 are the base area, the number of edges in the base, and the solid angle at the apex respectively of a pyramid of unit height; and $P_2, m_2,$ and ϕ_2 are the corresponding quantities of another pyramid of unit height, then if we set $m = \frac{1}{2}(m_1 + m_2)$ and $\phi = \frac{1}{2}(\phi_1 + \phi_2)$, then the following relation holds

$$2P(m, \phi) \leq P(m_1, \phi_1) + P(m_2, \phi_2) \leq P_1 + P_2.$$

Note that the lemma is not restricted to regular pyramids. The foregoing set of inequalities shows that the transition from non-regular to regular bases, and then to a pair of equal regular average polygons, reduces the sum of the areas of the bases.

5. *A Closer Lower Bound,—The Hexagon Ideal.* Since, for any real polyhedron, the average polygon has fewer edges than a hexagon, (see Sec. 3), a closer bound than the Circle Ideal can be obtained by using regular hexagons instead of circles. Using the pyramid function, substituting 6 for m , and $4\pi/n$ for ϕ , gives

$$P(m, \phi) = 6 \tan \frac{\pi}{6} \left[\sin^2 \frac{\pi}{6} \csc^2 \frac{\pi(n-2)}{6n} - 1 \right] \\ = 2\sqrt{3} \left[\frac{1}{4} \csc^2 \frac{\pi(n-2)}{6n} - 1 \right].$$

But $F = nP$, and $K = 9F$. Using $K(H)$ for the value of K thus obtained from the Hexagon Ideal,

$$K(H) = 9nP = 18\sqrt{3}n \left[\frac{1}{4} \csc^2 \frac{\pi(n-2)}{6n} - 1 \right].$$

6. *A Still Closer Lower Bound,—The Average Polygon Ideal.* However, since the greatest value that m can possibly attain is $6 - 12/n$, we can, by use of that value, obtain a still closer lower bound. The polygons may now be fictitious ones bounded by a fractional number of edges. Again, using the pyramid function, substituting $6 - 12/n$ for m , and $4\pi/n$ for ϕ , and using $K(A)$ for the value of K thus obtained from the Average Polygon Ideal,

$$K(A) = 9nP = 54(n-2) \tan E (4 \sin^2 E - 1),$$

where

$$E = \frac{\pi n}{6(n-2)}.$$

We prove the foregoing indicated theorem by successive application of the lemma to a real polyhedron obtaining the Average Polygon Ideal as a limit. We can now state the

Theorem. The value of $K(=F^3/V^2)$ for any real polyhedron of n faces is never less than the value of $K(A)$ obtained from the Average Polygon Ideal.

7. *Polyhedra which satisfy the Average Polygon Ideal.* In general, m is fractional. However, when $n=4, 6$, or 12 , it becomes an integer. These values correspond to three real polyhedra. Hence we have the

Corollary. The regular tetrahedron, the cube, and the regular dodecahedron have the greatest volume for a given surface among all the polyhedra possessing 4, 6, or 12 faces respectively.

Hence, to the list of known "best" polyhedra may be added the regular dodecahedron. We now have solutions for $n=4, 5, 6, 7$, and 12 .

It is not at all obvious that the regular polyhedra are "best" polyhedra. It is not difficult to show by elementary methods that the regular tetrahedron is "best". But the cube has been shown to be "best" only by very elaborate and difficult analysis. The regular octahedron and the regular icosahedron are *not* "best". Their comparison with "better" polyhedra is made in Table 2. Since each is bounded by only triangles, the foregoing discussion indicates that they are probably "very poor" polyhedra. The Theorem of the Average Polygon Ideal verifies the previously proved results that the regular tetrahedron and cube are "best", and it adds the regular dodecahedron. That the additional results are few is not surprising. Rather, it is gratifying that *any* concrete results could be obtained by such simple means.

8. *Conjectures based on the Average Polygon Ideal.* A necessary and sufficient condition for the equality to hold in the relation $m \leq 6 - 12/n$ is that all the vertices of the polyhedron be trihedral; and in this case the polyhedron possesses the maximum possible number of edges. If m is not an integer, no real polyhedron exists which has m edges for each polygon. This suggests that the "best" polyhedra, when m is fractional, may be found among those which approximate the Average Polygon Ideal by possessing, at most, only b -gons and $(b+1)$ -gons, and for which $m = 6 - 12/n$. All the known "best" polyhedra are of this class. Let us designate the polyhedra of this class as *medial polyhedra*.

9. *The Medial Polyhedra.* We may tabulate the types of polygons which compose the medial polyhedra as in the following page.

It is easy to show that no medial polyhedra exist for $n=11$ and for $n=13$. Except for these, there is exactly one for each n up to $n=15$. For $n=16$ there are at least two. The number of medial polyhedra increases quite rapidly after $n=16$, and there⁽¹⁾ are over 80 for $n=24$.

(1) Kirkman, T. P., Liverpool, Proc. Lit. Philos. Soc. vol. XXXVII (1882), p. 65.

Table 2.

The Isoperimetric Problem for Polyhedra.

n	Net	Description	Symbols	Calculated Angles (Inclinations to Axis)	K	$K(A)$	$\frac{K-}{K(A)}$
4	IV	Regular Tetra.	1, 3	—	374.12	374.12	0
"	"	"	2, 2	—	"	"	"
5	V	Prism	1, 3, 1	—	280.59	260.11	20.48
"	"	"	1, 2, 2	—	"	"	"
6	VI	Cube	1, 4, 1	—	216.0	216.0	0
"	"	"	2, 2, 2	—	"	"	"
"	"	"	3, 3	—	"	"	"
7	VII	Prism	1, 5, 1	—	196.17	192.29	3.88
"	"	"	1, 2, 2, 2	—	"	"	"
8	VIII		2, 2, 2, 2	53° -07', 15° -23'	180.23	177.45	2.78
9	IX		3, 3, 3	43° -40'	169.10	167.37	1.83
"	"		1, (4), 2, 2		"	"	"
10	X	Trunc. Double Skew Pyr.	1, 4, 4, 1	26° -50'	161.29	159.90	1.39
12	XII	or Reg. Dodeca.	1, 5, 5, 1	26° -34'	149.87	149.87	0
"	"	"	3, 3, 3, 3		"	"	"
"	"	"	2, 2, (4), 2, 2		"	"	"
14	XIV	Trunc. Double Skew Pyr.	1, 6, 6, 1	26° -50'	143.89	143.37	0.52
15	XV		3, 3, 3, 3, 3			140.84	
"	"		1, (4), 2, (4), 2, 2			"	
16	XVI-1		4[1, 3]		139.26	138.82	0.44
"	"		1, (6), 3, 3, 3		"	"	"
16	XVI-2		3, 2, (2, 2, 2, 2), 2, 2			138.82	
17	XVII		1, 5, 5, 5, 1	52° -46' -40''	137.74	137.01	0.73
20	XX-1		1, 3, 3, (6), 3, 3, 1			132.87	
20	XX-2		1, 6, 6, 6, 1	51° -10'	133.31	132.87	0.44
"	"		3, 2, (4), (2, 2), (4), 2, 2		"	"	"
"	"		1, (4), 2, (4, 2), 2, (4), 1		"	"	"
22	XXII		4[1, 3, 3/2]			130.82	
24	XXIV		4[3, 3]			129.16	
32	XXXII	Trunc. Icosa. Chamfered	12[1, 5/3]	—	124.86	124.77	0.09
42	XLII	Dodeca.	12[1, 5/2]	—	121.92	121.80	0.12
∞		Sphere	—	—	113.10	113.10	0
8		Reg. Octa.	—	—	187.06	177.45	9.61
8		Prism	1, 6, 1	—	187.06	177.45	9.61
8		Double Skew Pyr.	4, 4	35° -15' -52''	181.55	177.45	4.10
20		Reg. Icosa.	—	—	136.46	132.87	3.59

Table of Polyhedra which satisfy the Lindelöf Necessary Condition.

more than one is given for the same value of n , the Arabic numeral which follows distinguishes them from each other.

11. *Polyhedron Symbols.* Instead of using the net, one may describe the medial polyhedra (and also the skew double pyramids and several more general types) by means of a topological symbol. The symbol for XIV is 1, 6, 6, 1. This states that a face center is located at each of the two poles of the sphere and the remaining face centers lie in two rings of six centers each. All of the faces of a ring are congruent.

Sometimes a polyhedron may be described by each of several symbols. Table 2 gives all of the symbols for each of the presented medial polyhedra up to $n=20$. Beyond $n=20$, only one symbol is given. Likewise, in Fig. 1, only a sample symbol is given.

The regular dodecahedron has three symbols, namely, 1, 5, 5, 5, 1; 3, 3, 3, 3; and 2, 2, (4), 2, 2. In each of the foregoing symbols, with the exception of the last, the centers are uniformly distributed around the rings; and the meridians of the centers in each ring lie midway between the meridians of the centers of the adjacent rings. In the last symbol, (4) indicates four congruent faces whose centers are not uniformly distributed around the ring. Similarly, whenever numbers are placed in parenthesis, it indicates a departure from uniform distribution, and sometimes even a departure from congruence. The latter condition is shown in the symbol 1, (4), 2, (4, 2), 2, (4), 1 for XX-2 in which the equatorial ring possesses four pentagons and two hexagons.

In XVI-1, the symbol 4[1, 3] indicates four points of symmetry, at each point a polygon surrounded by a ring of three polygons. In XXXII, the symbol 12[1, 5/3] indicates twelve points of symmetry, at each point a pentagon surrounded by five hexagons, but each of the five hexagons is common to three of the twelve patches. The author hopes to develop this symbolism and the related polyhedra more fully in a later paper.

12. *Calculations.* By choosing the proper inclinations of the successive rings or zones, one may determine the "best" medial polyhedra of any particular type. The Lindelöf condition suffices for this purpose. The center of gravity of each face may be made to coincide with the point of contact of the face with the sphere by the solution of an equation or, if the equation is too cumbersome, by successive approximation. Table 2 gives some of the inclinations

and the corresponding values of K . The comparison with $K(A)$ (from the Average Polygon Ideal for the same number of faces) is made and the excess noted. For large values of n , the value $K(A)$ may serve as a close approximation of the value of K for "best" polyhedra.

For purposes of investigation, it is fortunate that the medial polyhedra up to $n=15$, and for many values of n greater than 15, possess sufficient symmetry to enable calculation. In VIII, and likewise in XV, the formula for K can be reduced to a function of two angles of inclination; while in IX, X, XIV, XVI, XVII, and XX-2, one angle is sufficient.

It is conceivable that, for some large value of n , there might exist a medial polyhedron which does not possess a single pair of congruent faces. The spacial orientations of the faces would then be all different and the arrangement would not possess symmetry. In such a case the number of variables would be so great that it would be hopeless to attempt the prodigious task of solving for the spacial orientation of every face and calculating the value of K with sufficient accuracy to compare it with another medial polyhedron of the same number of faces.

The medial polyhedra in Table 2 for n greater than 15 are chosen for their symmetry, and hence, for the ease of their calculation. However, the medial polyhedra up to $n=15$ are unique. If the proposition is true, then they are "best". It may be that they are "best" even should the proposition fail for large values of n .

That the proposition does give valid clues is exemplified in the case of the medial polyhedron VIII. It has not been considered previously in the literature as a possibility for the "best" octahedron. Steinitz offered the double skew pyramid⁽¹⁾. But VIII is appreciably "better", and in view of the small number of likely octahedra, all of which have been considered, it is probably the "best" although a rigorous proof is not given here. The difficulties associated with a complete proof, which would involve the examination of every octahedron, have been already discussed in this paper.

(1) Steinitz gives the incorrect value of 173.40 for K .

On the Summability of the Derived Fourier Series by Riesz's Logarithmic Means,

by

Fu Traing WANG, Sendai.

1. Let $f(t)$ be summable and periodic with period 2π , and let

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

Differentiating the series termwise, we get the series

$$\sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt), \quad (1.1)$$

which is called the derived Fourier series of $f(t)$.

If $\frac{f(x+t) - f(x-t)}{t}$ is summable in $(0, \delta)$ and there is a constant A such that

$$\int_0^t \left\{ \frac{f(x+t) - f(x-t)}{2t} - A \right\} dt = o(t) \quad (1.2)$$

as $t \rightarrow 0$, then the derived Fourier series (1.1) is summable (C, k) ($k > 2$) to sum A , for $t = x^{(1)}$.

This theorem is not true, when $k=2$. For Prof. Hahn⁽²⁾ proved that there is an even function $\phi(t)$ such that

$$\int_0^t \phi(u) du = o(t),$$

and the Fourier series of $\phi(t)$ is not summable $(C, 1)$ for $t=0$. The derived Fourier series of $t\phi(t)$ is not summable $(C, 2)$ for $t=0$, however the condition (1.2)

$$\int_0^t \frac{u\phi(u) - (-u)\phi(-u)}{2u} du = \int_0^t \phi(u) du = o(t)$$

is satisfied.

Let $k > 0$, and

$$B_{\omega}^k = \frac{1}{(\log \omega)^k} \sum_{n < \omega} \left(\log \frac{\omega}{n} \right)^k n(b_n \cos nx - a_n \sin nx).$$

(1) F. T. Wang, This Journal, vol. 40 (1934), pp. 142-159.

(2) H. Hahn, Jahresb. der Deutsch. Math. Vereinig., vol. 25 (1916).